22. ARITMETIČKI I GEOMETRIJSKI NIZ

① niz a_n

$a_1 = 1$

$a_2 = 2$

$n > 2 \quad a_{n+2} = \dfrac{a_{n+1}}{a_n}$

$a_{777} = ?$

članovi niza: $\underbrace{\underbrace{1, 2, 2}_{\text{odbrojimo od početka 3. član}}, 1, \frac{1}{2}, \frac{1}{2}}_{6 \text{ članova}}, \ldots$ ponavlja se

$\dfrac{777}{6} = 129$

$129 \cdot 6 = 774$

$777 - 774 = 3$ ostatak

$$\boxed{a_{777} = 2}$$

② (Pr 5)

aritmetički niz

opći član arit. niza $\quad a_n = a_1 + (n-1)d$

$a_5 + a_9 = 11 \quad (1)$

$a_5 \cdot a_9 = 28 \quad (2)$

$\left.\begin{aligned} a_5 &= a_1 + (5-1)d = a_1 + 4d \\ a_9 &= a_1 + (9-1)d = a_1 + 8d \end{aligned}\right\} (1), (2)$

$a_1 + 4d + a_1 + 8d = 11 \quad (3) \Rightarrow 2a_1 + 12d = 11 \Rightarrow a_1 = \frac{11}{2} - 6d \rightarrow (4)$

$(a_1 + 4d) \cdot (a_1 + 8d) = 28 \quad (4)$

$\left(\frac{11}{2} - 6d + 4d\right) \cdot \left(\frac{11}{2} - 6d + 8d\right) = 28$

$\underbrace{\left(\frac{11}{2} - 2d\right) \cdot \left(\frac{11}{2} + 2d\right)}_{} = 28$

$\left(\frac{11}{2}\right)^2 - (2d)^2 = 28$

$-4d^2 = 28 - \left(\frac{11}{2}\right)^2 = 28 - \frac{121}{4} = \frac{112 - 121}{4} = -\frac{9}{4} \quad / : (-4)$

$d^2 = \frac{9}{16}$

$d = \frac{3}{4}$

$a_1 = \frac{11}{2} - \cancel{6}^{3} \cdot \frac{3}{\cancel{4}_{2}} = \frac{2}{2} = 1$

$$\boxed{a_n = 1 + \frac{3}{4}(n-1)}$$

③ (Pr 6)

zbroj svih troznamenkastih brojeva koji pri dijeljenju sa 5 imaju ostatak 2 ili 3.

102, 103, 107, 108, 112, 113, 117, 118, ...

grupiramo ih u 2 podniza

1. podniz – sa ostatkom 2 : 102, 107, 112, 117, ... 997

2. podniz – " – " – 3 : 103, 108, 113, 118, ... 998

broj članova : $\frac{997-102}{5}+1=\underline{180}=n_1=n_2$

Zbroj članova niza: $S=\frac{n}{2}(a_1+a_n)$

$\to S=S_1+S_2$

$S_1=\frac{n_1}{2}(a_{11}+a_{n1})=\frac{180}{2}(102+997)=90\cdot 1099$

$S_2=\frac{n_2}{2}(a_{12}+a_{n2})=\frac{180}{2}(103+998)=90\cdot 1101$

$(+)=\boxed{198\,000}$ $\boxed{=S}$

④ (Pr 8)

$a_n=\frac{1}{2}n-25$

$S_N=?$

$\to$ zbroj svih negativnih članova niza

negativni članovi;

$a_n<0$

$\frac{1}{2}n-25<0$

$\Rightarrow n<50$

$\underline{n=49}$

$n=1 \Rightarrow \underline{a_1}=\frac{1}{2}\cdot 1-25=\underline{-\frac{49}{2}}$

$n=49 \Rightarrow \underline{a_{49}}=\frac{1}{2}\cdot 49-25=\underline{-\frac{1}{2}}$

$\boxed{S_N=\frac{49}{2}\left(-\frac{49}{2}-\frac{1}{2}\right)=\frac{49}{2}\cdot(-25)=\underline{-612,5}}$

5 (Pr 11)

geometrijski niz $\Longrightarrow$ opći član:

$$a_n = a_1 \cdot q^{n-1}$$

$$q = \frac{a_n}{a_{n-1}}$$

$a_2 = 3$

$a_5 = 12$

$a_8 = ?$

I) $a_5 = a_2 \cdot q^3$

$a_8 = a_5 \cdot q^3$

$12 = 3 \cdot q^3 \Longrightarrow \underline{q^3 = 4}$

$\boxed{a_8 = 12 \cdot 4 = 48}$

II) $a_n^2 = a_{n-k} \cdot a_{n+k}$

$k = 3 \qquad a_5^2 = \underbrace{a_{5-3}}_{a_2} \cdot \underbrace{a_{5+3}}_{a_8} \Longrightarrow \boxed{a_8 = \frac{a_5^2}{a_2} = \frac{12^2}{3} = \frac{144}{3} = 48}$

6 (Pr 13)

geometrijski niz

$a_1 \cdot a_3 \cdot a_5 \cdot \ldots \cdot a_{31} = 625$

$a_{16} = ?$

$$a_m \cdot a_n = a_u \cdot a_v \Longleftrightarrow m + n = u + v$$

$$a_1 \cdot a_{31} = a_3 \cdot a_{29} = \ldots = a_{15} \cdot a_{17} = a_{16}^2$$

16 parova ukupno

$$a_{16}^{2^4} = 625 = 5^4$$

$$a_{16}^{16} = 5^4 \quad / ^{\frac{1}{16}}$$

$$\boxed{a_{16} = \left(5^{\cancel{4}^{1}}\right)^{\frac{1}{\cancel{16}_4}} = 5^{\frac{1}{4}} = \sqrt[4]{5}}$$

23. DERIVACIJE FUNKCIJA

① (Pr 2)

$$f(x) = \frac{2}{x} + 3\sqrt{x} = \frac{2}{x} + 3x^{\frac{1}{2}} = 2x^{-1} + 3x^{\frac{1}{2}} \quad /'$$

$$\left[\frac{d}{dx}\right] \quad '$$

$$\left(2x^{-1} + 3x^{\frac{1}{2}}\right)' = 2\cdot(-1)x^{-1-1} + 3\cdot\frac{1}{2}x^{\frac{1}{2}-1}$$

$$= -2x^{-2} + \frac{3}{2}x^{-\frac{1}{2}} = \boxed{\frac{-2}{x^2} + \frac{3}{2\sqrt{x}}}$$

② (Pr 3)

a) $$f(x) = (x+1)\overbrace{\sqrt{x}}^{x^{\frac{1}{2}}} + \overset{x^{-1}}{\frac{1}{x}} \quad /'$$

$$f'(x) = \frac{df(x)}{dx} = \left[(x+1)\sqrt{x} + \frac{1}{x}\right]' = \left[(x+1)'\sqrt{x} + (x+1)(\sqrt{x})' + \left(\frac{1}{x}\right)'\right] = 1\cdot\sqrt{x} + (x+1)\frac{1}{2}x^{-\frac{1}{2}} + (-1)x^{-2}$$

$$= \sqrt{x} + \frac{1}{2}(x+1)\frac{1}{\sqrt{x}} - \frac{1}{x^2}$$

$$= \sqrt{x} + \left(\frac{1}{2}x + \frac{1}{2}\right)\frac{1}{\sqrt{x}} - \frac{1}{x^2} = \sqrt{x} + \frac{\overset{\sqrt{x}}{x}}{2\sqrt{x}} + \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$$

$$\boxed{= \frac{3}{2}\sqrt{x} + \frac{1}{2\sqrt{x}} - \frac{1}{x^2}}$$

b) $$g(x) = x\cdot\sin x \quad /'$$

$$\boxed{g'(x) = x'\cdot\sin x + x\cdot(\sin x)' = \sin x + x\cdot\cos x}$$

③ (Pr 4)

$$f(x) = \sin\sqrt{1+x^2} \quad /'$$

$$\boxed{f'(x) =} \left(\sin\sqrt{1+x^2}\right)' = \cos\sqrt{1+x^2}\cdot\Big(\overbrace{\sqrt{1+x^2}}^{(1+x^2)^{\frac{1}{2}}}\Big)' =$$

$$= \cos\sqrt{1+x^2}\cdot\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot 2x$$

$$= \frac{\cancel{2}^{1} x \cdot \cos\sqrt{1+x^2}}{{}_1\cancel{2}\sqrt{1+x^2}} = \boxed{\frac{x \cdot \cos\sqrt{1+x^2}}{\sqrt{1+x^2}}}$$

④ (Pr 5)

$$f(x) = \ln\sqrt{x e^x} \quad /'$$

$$\boxed{f'(x)} = \left(\ln\sqrt{xe^x}\right)' = \frac{1}{\sqrt{xe^x}} \cdot \frac{1}{2}\left(xe^x\right)^{-\frac{1}{2}} \cdot \left(1 \cdot e^x + x \cdot e^x\right)$$

$$= \frac{1}{\sqrt{xe^x}} \cdot \frac{1}{2} \frac{1}{\sqrt{xe^x}} \cdot \left(e^x + x \cdot e^x\right)$$

$$= \frac{1}{2x\cancel{e^x}_1} \cancel{e^x}^1 (1+x) = \boxed{\frac{x+1}{2x}}$$

⑤ (Pr 6)

$$f(x) = \frac{\sin x - \cos x}{\sin x + \cos x} \quad /'$$

$$f'(x) = \frac{(\sin x - \cos x)'(\sin x + \cos x) - (\sin x - \cos x) \cdot (\sin x + \cos x)'}{(\sin x + \cos x)^2}$$

$$= \frac{(\cos x + \sin x)(\sin x + \cos x) - (\sin x - \cos x)(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= \frac{(\sin x + \cos x)^2 + (\cos x - \sin x)^2}{(\sin x + \cos x)^2} = \frac{2\overbrace{(\sin^2 x + \cos^2 x)}^{=1}}{\underbrace{\sin^2 x + 2\sin x\cos x + \cos^2 x}_{=1}}$$

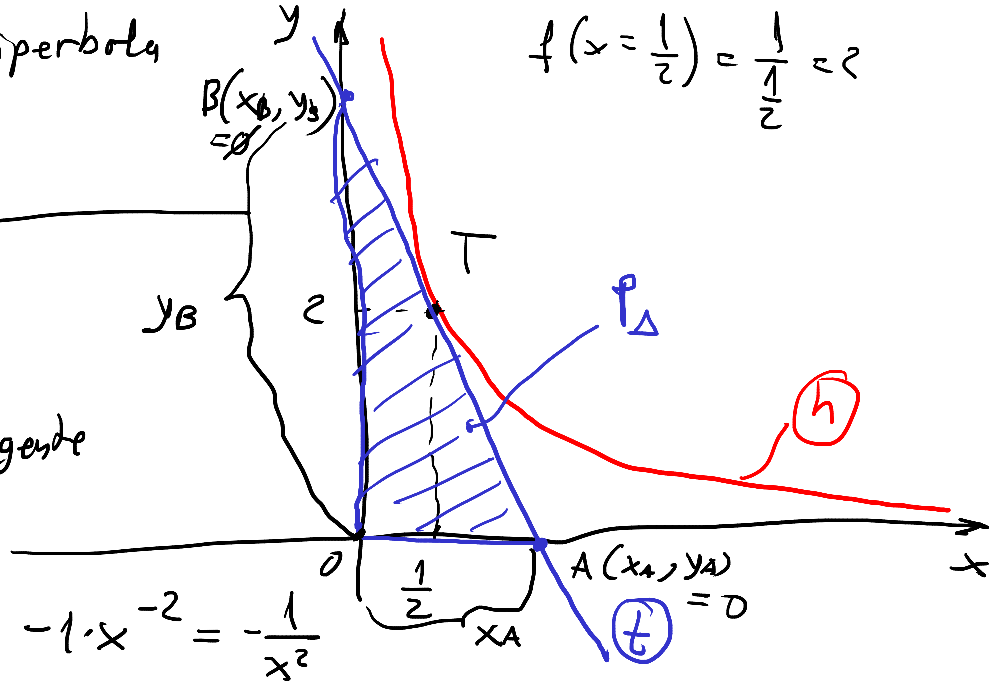
$$= \boxed{\frac{2}{1 + 2\sin x \cos x}}$$

(6) (h) ... $f(x) = \frac{1}{x}$ – hiperbola

$f\left(x=\frac{1}{2}\right) = \frac{1}{\frac{1}{2}} = 2$

$x_0 = \frac{1}{2}$

$P_\Delta = ?$



koeficijent smjera tangente

$f'(x)$

$$f'(x) = \left(\frac{1}{x}\right)' = (x^{-1})' = -1 \cdot x^{-2} = -\frac{1}{x^2}$$

$$f'\left(x=\frac{1}{2}\right) = -\frac{1}{\left(\frac{1}{2}\right)^2} = -\frac{1}{\frac{1}{4}} = -4 = a_t$$

$T\left(\frac{1}{2}, 2\right)$

jednadžba pravca kroz točku: $y - y_0 = \underbrace{f'(x_0)}_{a_t} \cdot (x - x_0)$

$$y - 2 = -4\left(x - \frac{1}{2}\right)$$

$$y - 2 = -4x + 2$$

$$y = -4x + 4 \quad \dots (t)$$

točke A i B

$y = 0 = -4x + 4 \Rightarrow x = 1 = x_A$

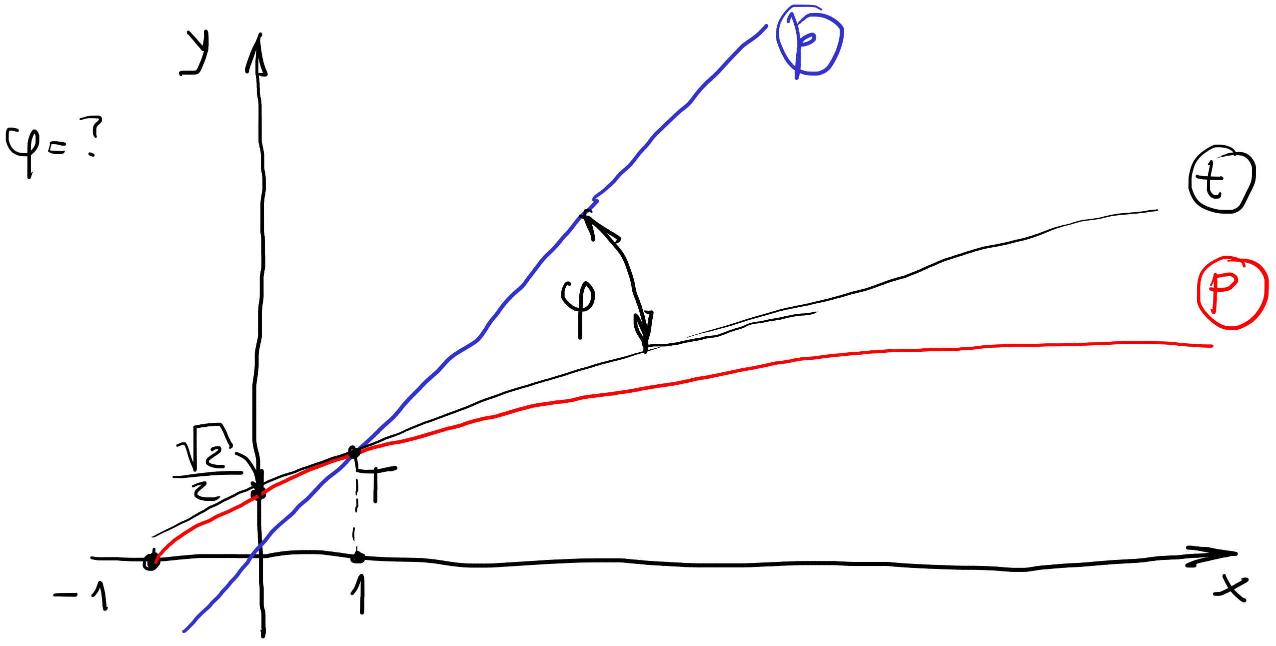
$y = -4 \cdot 0 + 4 = 4 = y_B$

$$\boxed{P_\Delta = \frac{1}{2} \cdot x_A \cdot y_B = \frac{1}{2} \cdot 1 \cdot 4 = 2}$$

(7) (Pr 8)

(p) ... $y = x$

(P) ... $y = \sqrt{\frac{1}{2}x + \frac{1}{2}}$ parabola uzduž x-osi



<u>točka T:</u>

$$y_p = y_P$$

$$x = \sqrt{\frac{1}{2}x + \frac{1}{2}} \quad /^2$$

$$x^2 - \frac{1}{2}x - \frac{1}{2} = 0 \quad \begin{cases} x_1 = -\frac{1}{2} & \ominus \\ x_2 = 1 \end{cases}$$

$$y(x=1) = 1 \qquad \underline{T(1,1)}$$

(p) $y = x \;/'\qquad y' = 1 = k_p$

(t) $y - y_T = f'(x) \cdot (x - x_T)$

$$f'(x) = \left(\sqrt{\frac{1}{2}x + \frac{1}{2}}\right)' = \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{1}{2}x + \frac{1}{2}}} \cdot \frac{1}{2} = \frac{1}{4\sqrt{\frac{1}{2}x + \frac{1}{2}}}$$

$$f'(x_T = 1) = \frac{1}{4\sqrt{\frac{1}{2} \cdot 1 + \frac{1}{2}}} = \frac{1}{4} = k_t$$

φ – kut između pravaca

$$\operatorname{tg}\varphi = \left|\frac{k_1 - k_2}{1 + k_1 k_2}\right| = \left|\frac{1 - \frac{1}{4}}{1 + 1 \cdot \frac{1}{4}}\right| = \left|\frac{\frac{3}{4}}{\frac{5}{4}}\right| = \frac{3}{5} = 0{,}6$$

$$\Rightarrow \boxed{\varphi = \operatorname{arctg}(0{,}6) = 30{,}964^\circ = 30^\circ 57' 49''}$$

8 (zadaci - 14)

$$f(x) = x^2 \ln x + \sin 2x \;/'$$

$$f'(x) = 2x \cdot \ln x + x^2 \cdot \frac{1}{x} + 2\cos 2x = 2x \ln x + x + 2\cos 2x \;/'$$

$$\boxed{f''(x)} = 2\ln x + 2\cancel{x} \cdot \frac{1}{\cancel{x}} + 1 + 2(-\sin 2x) \cdot 2$$

$$= 2\ln x + 2 + 1 - 4\sin 2x = \boxed{2\ln x - 4\sin 2x + 3}$$

9 (zadaci - 24)

$$f(x) = x^3 + 3x^2 + 9x - 12$$

$p \;\dots\; y = 6x \Rightarrow k_p = 6$

$t \parallel p \quad k_t = 6$

$y_t = ?$

$$\left.\begin{array}{l} f'(x) = k_t = 3x^2 + 6x + 9 \\ k_t = 6 \end{array}\right\} (=)$$

$$3x^2 + 6x + 9 = 6 \;/ : 3$$

$$x^2 + 2x + 3 - 2 = 0$$

$$x^2 + 2x + 1 = 0$$

$$(x+1)^2 = 0 \Rightarrow x_{1,2} = -1$$

$t \;\dots\; y_t = 6x + b_t$?

za $x = -1$ $\quad f(-1) = (-1)^3 + 3 \cdot (-1)^2 + 9(-1) - 12 = -19$

$(-1, -19)$

$$-19 = 6 \cdot (-1) + b_t$$

$$b_t = -13$$

$$\boxed{y_t = 6x - 13}$$

(10) (Ispit 1 – 3)

$$\left.\begin{array}{l} f(x) = 2x^2 - 3 \quad \cdots\cdots \; t_1 \\ g(x) = \ln(3x) + 2 \quad \cdots \; t_2 \end{array}\right\} t_1 \parallel t_2$$

$x = ?$

$$\left.\begin{array}{l} f'(x) = 4x = k_{t_1} \\ g'(x) = \frac{1}{3x} \cdot 3 = \frac{1}{x} = k_{t_2} \end{array}\right\} (=)$$

jer su t_1 i t_2 paralelne

$$4x = \frac{1}{x} \;/\cdot x$$

$$x^2 = \frac{1}{4}$$

$$\boxed{x = \frac{1}{2}}$$

(11) (Ispit 2 – 3)

$$f(x) = \operatorname{tg} x + \frac{1}{\cos x}$$

$$\boxed{f'(x)} = \frac{1}{\cos^2 x} + \left((\cos x)^{-1}\right)' = \frac{1}{\cos^2 x} - (\cos x)^{-2} \cdot (-\sin x)$$

$$= \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} = \boxed{\frac{1 + \sin x}{\cos^2 x}}$$

24. PRIMJENE DIFERENCIJALNOG RAČUNA

1 (Pr 3)

$$f(x)=\frac{1}{3}x^3-2x^2+3x$$

ekstremi funkcije:

$$f'(x)=0$$

$$f'(x)=\frac{1}{3}\cdot 3x^2-2\cdot 2x+3=x^2-4x+3$$

$$x^2-4x+3=0 \Rightarrow \left.\begin{matrix} x_1=1 \\ x_2=3 \end{matrix}\right\}$$ stacionarne točke

$$f''(x)=2x-4$$

$$f''(x=1)=2\cdot 1-4=2-4=-2<0$$ maksimum

$$f''(x=3)=2\cdot 3-4=2>0$$ minimum

$$f(x=1)=\frac{1}{3}\cdot 1^3-2\cdot 1^2+3\cdot 1=\frac{4}{3}$$

$$f(x=3)=\frac{1}{3}\cdot 3^3-2\cdot 3^2+3\cdot 3=0$$

ekstremi:

$$\boxed{\begin{matrix} T_1\left(1,\frac{4}{3}\right) \\ T_2(3,0) \end{matrix}}$$

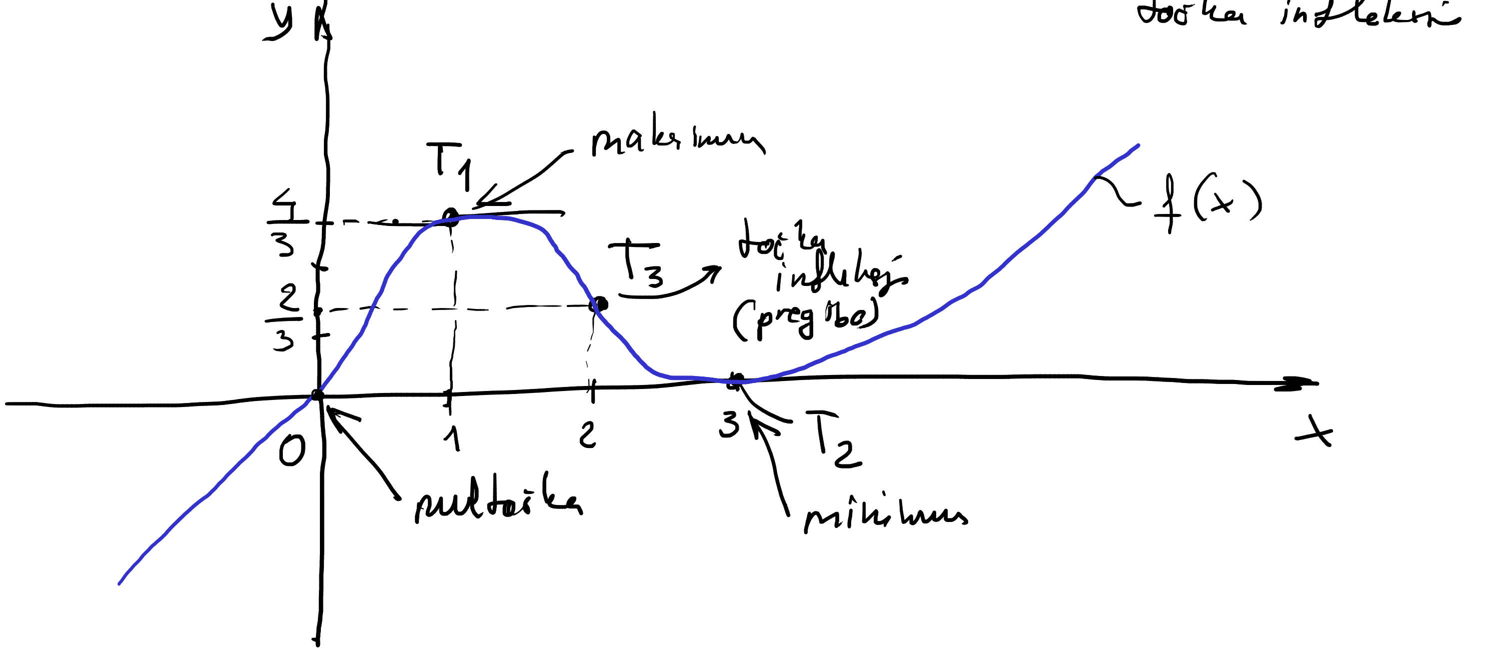
$$f''(x)=0$$

$$2x-4=0 \Rightarrow x=2$$

$$f(x=2)=\frac{1}{3}\cdot 2^3-2\cdot 2^2+3\cdot 2=\frac{2}{3}$$

$$\boxed{T_3\left(2,\frac{2}{3}\right)}$$

točka infleksije



intervali	$f'(x)$	ponašanje funkcije
$\langle -\infty, 1 \rangle$	>0	$\nearrow$ raste
$\langle 1, 3 \rangle$	<0	$\searrow$ pada
$\langle 3, +\infty \rangle$	>0	$\nearrow$ raste

(2) (Pr 4)

$$f(x) = (x^3 - x^2)\, e^x$$

Nultočke funkcije

$$f(x) = 0$$

$$(x^3 - x^2)\, e^x = 0$$

$$x_{1,2} = 0 \qquad y_{1,2} = 0$$

$$x_3 = 1 \qquad y_3 = 0$$

$$\boxed{\begin{array}{l} T_{1,2}(0,0) \\ T_3(1,0) \end{array}}$$

Stacionarne točke:

$$f'(x) = (3x^2 - 2x)e^x + (x^3 - x^2)e^x$$

$$= (x^3 + 2x^2 - 2x)\, e^x$$

$$f'(x) = 0$$

$$(x^3 + 2x^2 - 2x)\, e^x = 0 \quad / : e^x$$

$$x(x^2 + 2x - 2) = 0$$

$$x_1 = 0$$

$$x_{2,3} = -1 \pm \sqrt{3}$$

$$f''(x) = (3x^2 + 4x - 2)e^x + (x^3 + 2x^2 - 2x)\, e^x$$

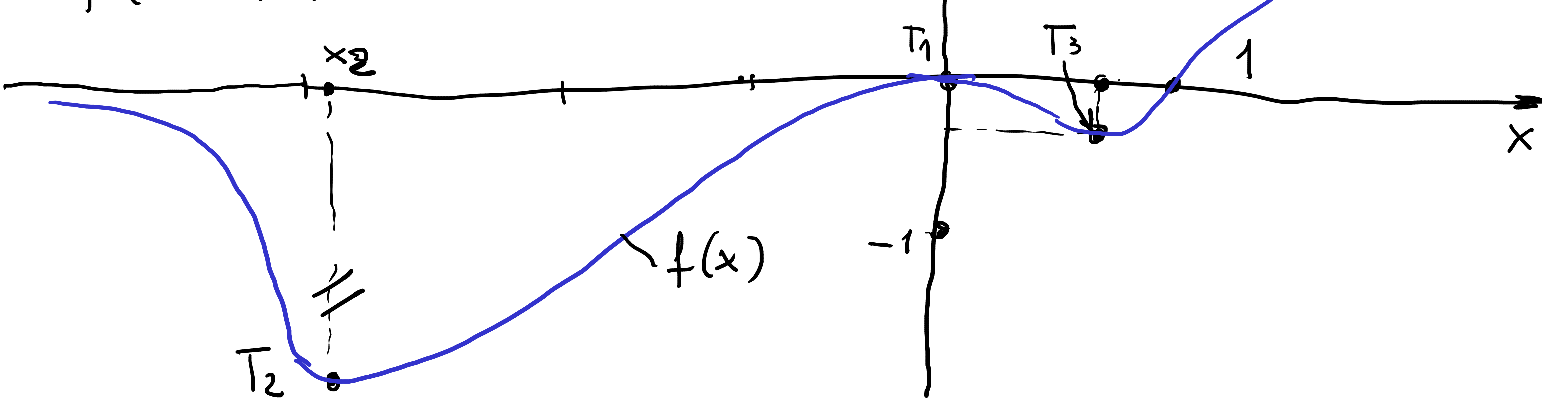
$$= (x^3 + 5x^2 + 2x - 2)\, e^x$$

x	$f(x)$
$-1-\sqrt{3}$	$\approx -427,97$
$-1+\sqrt{3}$	$\approx -0,3$

$f''(0) = -2 < 0$ lokalni maksimum

$f''(-1+\sqrt{3}) > 0$ lokalni minimum

$f''(-1-\sqrt{3}) > 0$ lokalni minimum



interval	ponašanje funkcije
$\langle -\infty, x_2 \rangle$	↘ pada
$\langle x_2, 0 \rangle$	↗ raste
$\langle 0, x_3 \rangle$	↘ pada
$\langle x_3, 1 \rangle$	↗ raste
$\langle 1, +\infty \rangle$	↗ raste

točke infleksije: $f''(x) = 0$

3 (Pr 6)

skup pravokutnih trokuta

$c = \sqrt{2}$ cm

$O_{max} = ?$

Pitagorin poučak: $a^2 + b^2 = c^2$

$$a^2 + b^2 = (\sqrt{2})^2 = 2 \Rightarrow a = \sqrt{2 - b^2}$$

$$O = a + b + \underbrace{c}_{=\sqrt{2}}$$

$$O = \sqrt{2 - b^2} + b + \sqrt{2} = f(b) \;/\; \frac{dO}{db}$$

$$\frac{dO}{db} = \frac{1}{2}(2 - b^2)^{-\frac{1}{2}} \cdot (-2b) + 1 = \frac{-b}{\sqrt{2 - b^2}} + 1 = 0$$

$$\frac{-b}{\sqrt{2 - b^2}} = -1 \;/^2$$

$$\frac{b^2}{2 - b^2} = 1 \;/ \cdot (2 - b^2)$$

$$b^2 = 2 - b^2$$

$$2b^2 = 2 \Rightarrow b = 1 \text{ cm}$$

Provjera:

$$\frac{d^2O}{db^2} = \ldots = -\frac{2}{2-b}$$

$$O''(b = 1) = -\frac{2}{2-1} = -2 < 0 \quad \text{maksimum} \checkmark$$

$$a = \sqrt{2 - b^2} = \sqrt{2 - 1^2} = 1 \text{ cm}$$

$$\boxed{\begin{aligned} a &= 1 \text{ cm} \\ b &= 1 \text{ cm} \\ c &= \sqrt{2} \text{ cm} \end{aligned}}$$

④ Valjak

$V = 1000\ cm^3$

$P_{min} = ?$

$$V = r^2 \pi v \Rightarrow v = \frac{V}{r^2 \pi} = \frac{1000}{r^2 \pi}$$

$$P = 2r^2\pi + 2r\pi v$$

$$P = 2r^2\pi + 2\cancel{r\pi} \cdot \frac{1000}{r^{\cancel{2}}\cancel{\pi}} = 2r^2\pi + \frac{2000}{r} = f(r)$$

$$\frac{dP}{dr} = P'(r) = 4r\pi - \frac{2000}{r^2} = 0 \Rightarrow r = \sqrt[3]{\frac{500}{\pi}} \approx 5{,}42\ cm$$

Provjera:

$$\frac{d^2P}{dr^2} = \ldots = 4\pi + \frac{4000}{r^3}$$

$$P''(r = 5{,}42\ cm) = 4\pi + \frac{4000}{(5{,}42)^3} = \ldots > 0 \quad \text{minimum} \checkmark$$

$$\left.\begin{array}{l} \boxed{v = \frac{1000}{(5{,}42)^2 \cdot \pi} = 10{,}84\ cm} \\ \boxed{r = 5{,}42\ cm} \end{array}\right\} \text{valjak koji ima } \underline{\text{min. oplošje}}$$

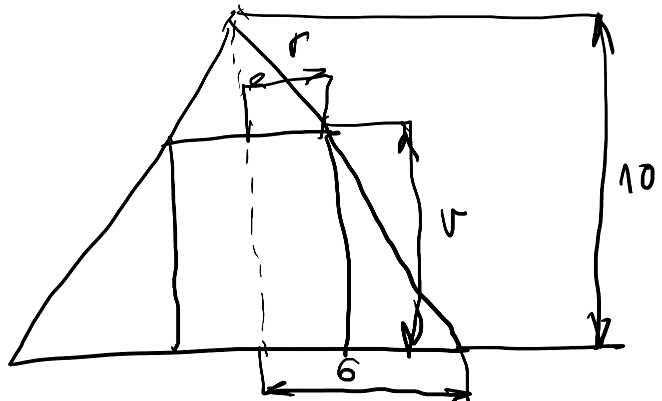
⑤ (Pr 8)

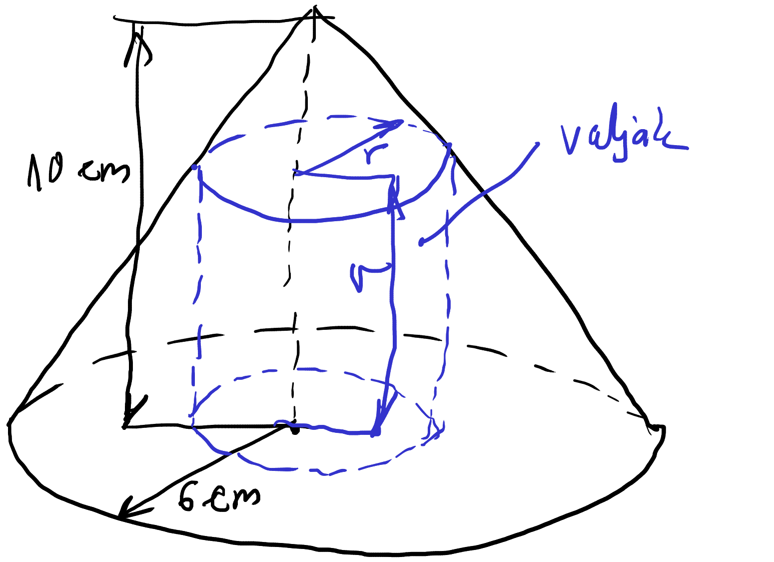
stožac

$r = 6\ cm$

$v = 10\ cm$

$V_{max,v} = ?$





sličnost trokuta:

$$\frac{10 - v}{r} = \frac{10}{6} \Rightarrow v = 10 - \frac{5}{3} r$$

$$V_v = r^2 \pi v = r^2 \pi \left(10 - \frac{5}{3} r\right) = 10 r^2 \pi - \frac{5}{3} r^3 \pi = f(r)$$

$$\frac{dV_v}{dr} = V_v'(r) = 20r\pi - \cancel{3}^{1} \cdot \frac{5}{\cancel{3}_{1}} r^2 \pi = 0$$

$$20r\pi = 5r^2\pi \quad / : 5r\pi$$

$$\boxed{r = 4}$$

$$\boxed{v = 10 - \frac{5}{3} r = 10 - \frac{5}{3} \cdot 4 = \frac{10}{3}}$$

Provjera:

$$\frac{d^2V_v}{dr^2} = 20\pi - 10r\pi$$

$$V_v''(r=4) = 20\pi - 10 \cdot 4 \cdot \pi = -20\pi < 0 \quad \text{maksimum} \checkmark$$