4. LINEARNE JEDNADŽBE I PROBLEMI PRVOG STUPNJA

① (Pr 1) $f(x) = -\frac{1}{2}x + \frac{3}{2}$

1) Nul-točka funkcije:

$f(x) = 0 = -\frac{1}{2}x + \frac{3}{2} \Rightarrow x = 3$ $\quad \boxed{T_1(3,0)}$

2) Promjena vrijednosti funkcije:

$x_1 = -4 \rightarrow x_2 = 5 \qquad \Delta f = f_2(x_2) - f_1(x_1)$

$f(x_1 = -4) = -\frac{1}{2}\cdot(-4) + \frac{3}{2} = \frac{7}{2}$

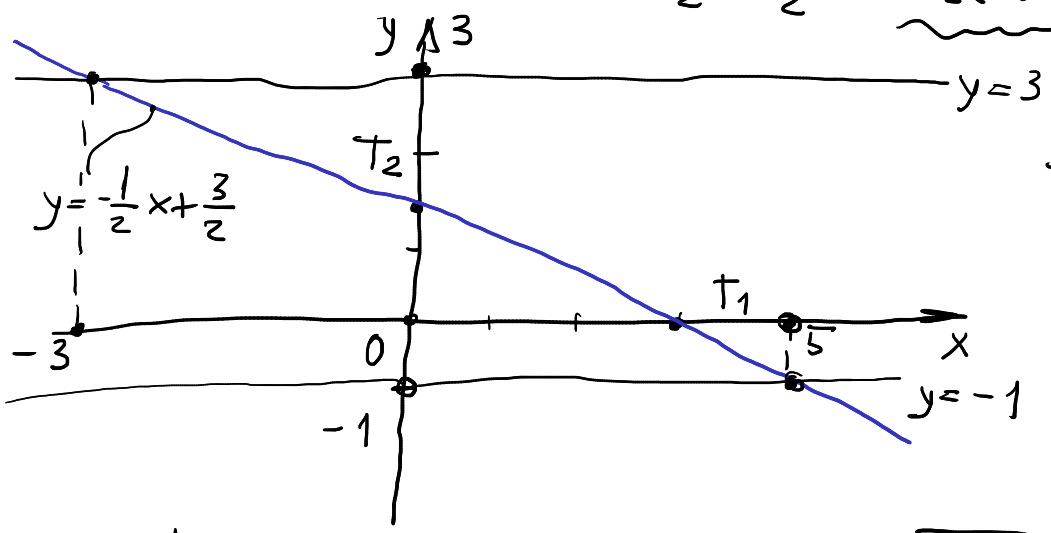
$f(x_2 = 5) = -\frac{1}{2}\cdot 5 + \frac{3}{2} = -\frac{5+3}{2} = -\frac{2}{2} = -1$ } Δf

$\boxed{\Delta f = -1 - \frac{7}{2} = \frac{-2-7}{2} = -\frac{9}{2}}$

3) $x = ?$

$-1 < f(x) \leq 3$

$x = 0 \Rightarrow f(x=0) = -\frac{1}{2}\cdot 0 + \frac{3}{2} = \frac{3}{2}$ $\quad T_2\left(0, \frac{3}{2}\right)$



$3 = -\frac{1}{2}x + \frac{3}{2}$

$-\frac{1}{2}x = 3 - \frac{3}{2}$

$-\frac{1}{2}x = \frac{3}{2}$

$x = -3$

$-1 = -\frac{1}{2}x + \frac{3}{2}$

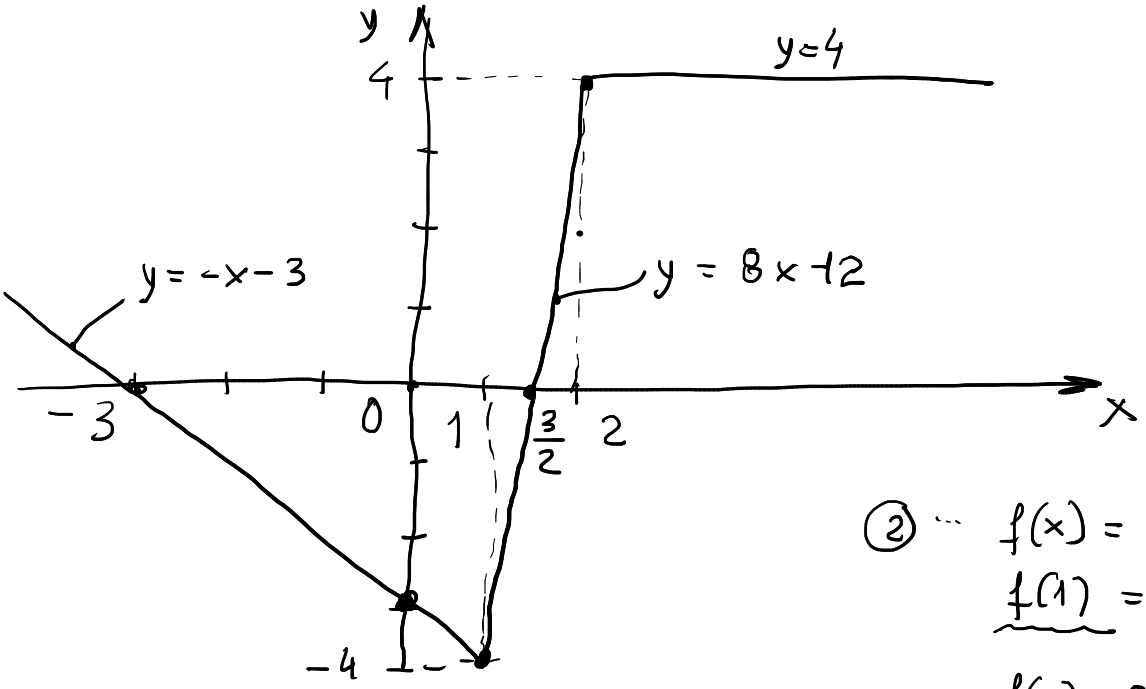
$-\frac{1}{2}x = -1 - \frac{3}{2} = \frac{-5}{2} \Rightarrow x = 5$

$\boxed{x \in [-3, 5\rangle}$

② (Pr 2)

$$f(x) = \begin{cases} -x-3, & x \leq 1 \\ 8x-12, & 1 < x \leq 2 \\ 4, & x \geq 2 \end{cases}$$

1) Grafični prikaz funkcij:



② ... $f(x) = 8x - 12$

$f(1) = 8 - 12 = -4$

$f(x) = 0 = 8x - 12$

$\Rightarrow x = \frac{12}{8} = \frac{3}{2}$

① ... $f(x) = -x - 3$

$f(x) = 0 = -x - 3 \Rightarrow x = -3$

$f(x=0) = 0 - 3 = -3$

2) $f(0) = -3$

$f(-11) = -(-11) - 3 = 8$

$f(101.11) = 4$

$f(-55.2) = -(-55.2) - 3 = 52.2$

$f(\sqrt{5}) = 4$

$f\left(\frac{5}{4}\right) = 8 \cdot \frac{5}{4} - 12 = -2$

③ (Pr 4)

linearna promjena: $f(x) = ax + b$

- x → broj prijeđenih kilometara [km]
- b → početna cijena [kn]
- a → cijena vožnje po jednom kilometru [kn/km]
- $f(x)$ → ukupna cijena vožnje [kn]

$f(x=7,5) = a \cdot 7,5 + b = 75 \quad / \cdot (-1)$ (1)

$f(x=11) = a \cdot 11 + b = 103$ (2)

$-7,5a - b = -75$

$11a + b = 103$ (+)

$3,5a = 28 \Rightarrow \boxed{a = 8 \text{ kn/km}} \rightarrow (2)$

$8 \cdot 11 + b = 103 \Rightarrow \boxed{b = 103 - 88 = 15 \text{ kn}}$

④ (Pr 5)

$f(x) = ax + b$

- x → broj dana [dan]
- b → masa plina [kg]

- kad se boca prazni $a < 0$ $\quad a = -0,3$ kg plina/dan
- kad je boca isprazna $f(x) = 0$

$b = 18$ kg plina - kad je boca puna!

1) $f(x) = 0 = ax + b = -0,3x + 18 \Rightarrow \boxed{x = \frac{18}{0,3} = 60 \text{ dana}}$

2) $f(x) = 15$ kg plina

$-0,3x + 18 = 15 \Rightarrow -0,3x = 15 - 18 = -3$

$\boxed{x = 10 \text{ dana}}$

⑤ (Pr 9)

$$\frac{1}{|2x-3|} + 8 = \frac{5}{|3-2x|}$$

$|2x-3| = |3-2x|$

$u = \frac{1}{|2x-3|} = \frac{1}{|3-2x|}$

$u + 8 = 5u$

$4u = 8$

$\Rightarrow u = 2$

$$\frac{1}{|2x-3|} = 2$$

$|2x-3| = \frac{1}{2} \Rightarrow$ dva rješenja:

1) $2x_1 - 3 = \frac{1}{2}$

$2x_1 = \frac{1}{2} + 3 = \frac{7}{2}$

$\boxed{x_1 = \frac{7}{4}}$

2) $-(2x_2 - 3) = \frac{1}{2}$

$-2x_2 = -\frac{5}{2}$

$\boxed{x_2 = \frac{5}{4}}$

(6)

$2x - 3y = 11 \quad (1)$

$4x + y = 1 \quad (2)$

3 načina: 1) metoda supstitucije
2) -||- suprotnih koeficijenata
3) Cramerovo pravilo

1) iz (2): $y = 1 - 4x \rightarrow (1)$

$2x - 3(1-4x) = 11$

$2x - 3 + 12x = 11$

$14x = 11 + 3 = 14$

$x = 1 \Rightarrow y = 1 - 4 \cdot 1 = 1 - 4 = -3$

2)

$2x - 3y = 11$
$4x + y = 1 \ / \cdot 3$ } (+)

$2x - 3y = 11$
$12x + 3y = 3$ } (+)

$14x = 14 \Rightarrow x = 1 \quad y = -3$

3) Cramerovo pravilo:

$2x - 3y = 11 \qquad ax + by = e$

$4x + y = 1 \qquad cx + dy = f$

$a = 2 \quad b = -3$

$c = 4 \quad d = 1$

$e = 11 \quad f = 1$

$$\boxed{x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ed - bf}{ad - cb} = \frac{11 \cdot 1 - (-3) \cdot 1}{2 \cdot 1 - 4 \cdot (-3)} = \frac{11 + 3}{2 + 12} = \frac{14}{14} = 1}$$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ce}{ad - cb} = \frac{2 \cdot 1 - 4 \cdot 11}{2 \cdot 1 - 4 \cdot (-3)} = \frac{2 - 44}{2 + 12} = \frac{-42}{14} = -3$$

⑦ (zadání -6)

$f(x) = ax + b \qquad f(x) = ?$

$f(-1) = -1$

$f(3) = 5$

$$\left.\begin{array}{l} a \cdot (-1) + b = -1 \;/\cdot(-1) \\ 3 \cdot a + b = 5 \end{array}\right\} (+)$$

$$\left.\begin{array}{l} a - b = 1 \\ 3a + b = 5 \end{array}\right\} (+)$$

$4a = 6 \Rightarrow \boxed{a = \frac{3}{2}}$

$\frac{3}{2} \cdot (-1) + b = -1$

$\boxed{b = -1 + \frac{3}{2} = \frac{1}{2}}$

$$\boxed{f(x) = \frac{3}{2}x + \frac{1}{2}}$$

1) Nul-body funkcji $f(x)$:

$f(x) = 0$

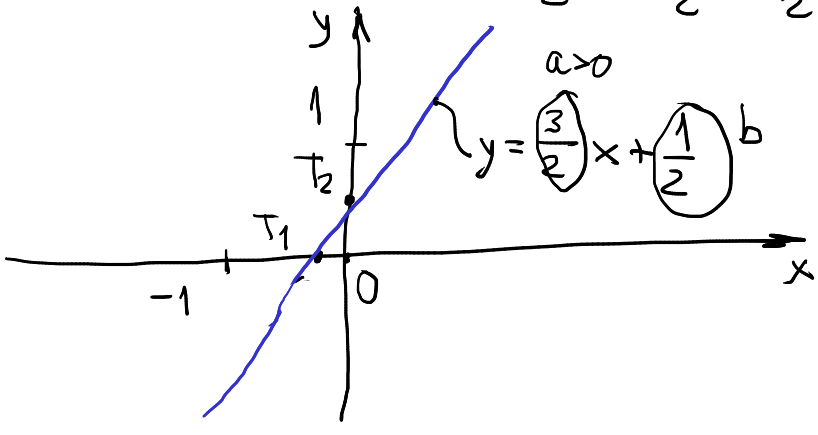
$\frac{3}{2}x + \frac{1}{2} = 0$

$\frac{3}{2}x = -\frac{1}{2} \;/\cdot\frac{2}{3} \Rightarrow x = -\frac{1}{3}$

$$\boxed{T_1\left(-\frac{1}{3}, 0\right)}$$

2) Graficky priklad funkcji $f(x)$:

$x = 0 \qquad f(x=0) = \frac{3}{2}x + \frac{1}{2} = \frac{3}{2} \cdot 0 + \frac{1}{2} = \frac{1}{2}$



$$\boxed{T_2\left(0, \frac{1}{2}\right)}$$

3) $-2 < f(x) \leq 4$

$x = ?$

$f(x) = -2 \qquad -2 = \frac{3}{2} \cdot x + \frac{1}{2} \Rightarrow x = -\frac{5}{3}$

$f(x) = 4 \qquad 4 = \frac{3}{2} \cdot x + \frac{1}{2} \Rightarrow x = \frac{7}{3}$

$\frac{3}{2}x + \frac{1}{2} = -2$

$\frac{3}{2}x = -2 - \frac{1}{2} = -\frac{5}{2} \quad / \cdot \frac{2}{3}$

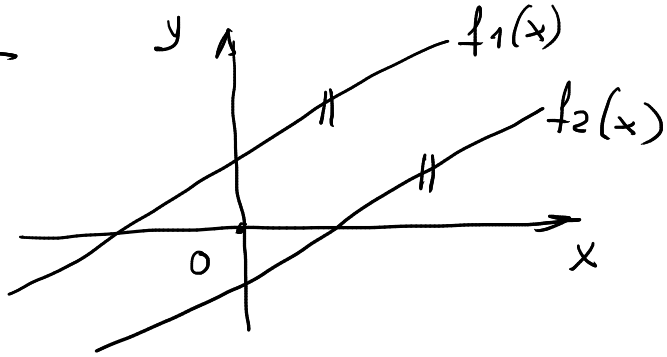
$x = -\frac{5}{3}$

$$-\frac{5}{3} < x \leq \frac{7}{3}$$

$$x \in \left\langle -\frac{5}{3}, \frac{7}{3} \right]$$

1) $a \neq 0$ - jedno rješenje

2) sustav nema rješenja:



3) beskonačno mnogo rješenja:

npr. $x - 2y = 1$

$2x - 4y = 2$

5. UREĐAJ NA SKUPU REALNIH BROJEVA

① (Pr 3)

$A \in \langle -\infty, 3]$

$B \in \langle -7, 5\rangle$

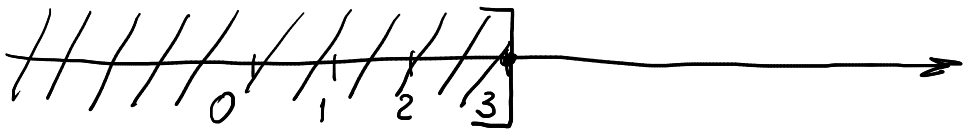
$C \in [-1, +\infty\rangle$

$A \cup B, B \cap C$ } ?

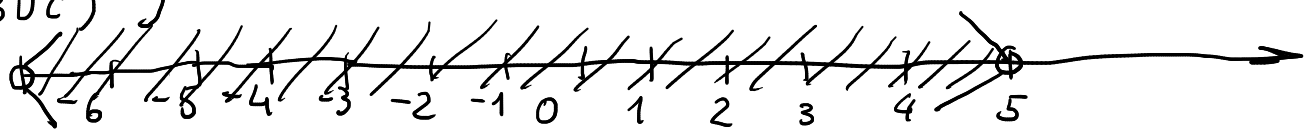
$A \cap (B \cup C)$

Skica intervala:

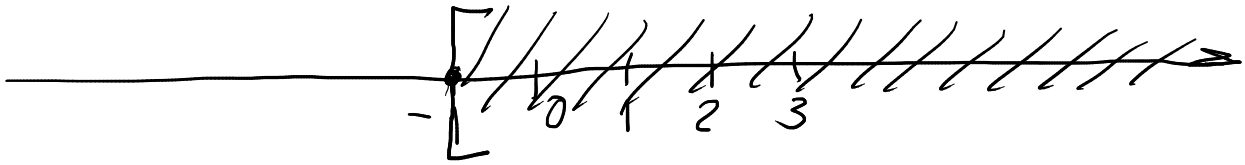
interval (A):



interval (B):



interval (C)



$A \cup B = \langle -\infty, 5\rangle$

$B \cap C = [-1, 5\rangle$

$A \cap (B \cup C) = ?$

$B \cup C = \langle -7, +\infty\rangle$

$\langle -\infty, 3] \cap \langle -7, +\infty\rangle = \langle -7, 3]$

② (Pr 4)

$$1 - \frac{x-0,5}{5} < \frac{3x}{2} - 0,1 \quad (1) / \cdot 10$$

$$\frac{4x-1}{3} - 0,25 > \frac{x}{4} \quad (2) / \cdot 12$$

$$10 - 2(x-0,5) < 5 \cdot 3x - 0,1 \cdot 10$$

$$10 - 2x + 1 < 15x - 1$$

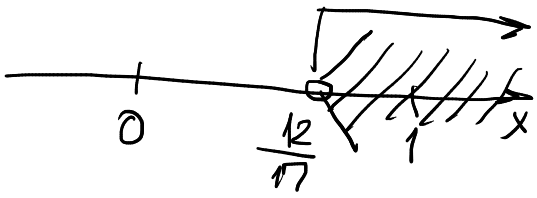
$$-2x - 15x < -1 - 10 - 1 = -12$$

$$-17x < -12 \ / : (-17)$$

↳ (!)

$x \in \left\langle \frac{12}{17}, +\infty \right\rangle$

$x > \frac{12}{17}$ → otvoreni interval



iz (2): $4(4x-1) - 0{,}25 \cdot 12 > 3x$

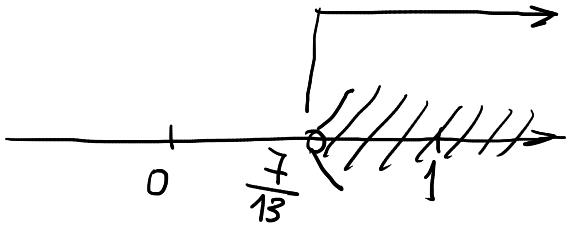
$16x - 4 - 3 > 3x$

$16x - 3x > 7$

$13x > 7$

$x > \frac{7}{13}$

$x \in \left\langle \frac{7}{13}, +\infty \right\rangle$



$\frac{12}{17} > \frac{7}{13}$

rješenje sustava nejednadžbi je presjek rješenja intervala:

$$\left\langle \frac{12}{17}, +\infty \right\rangle \cap \left\langle \frac{7}{13}, +\infty \right\rangle = \left\langle \frac{12}{17}, +\infty \right\rangle$$

(3) (Pr 5)

$$\frac{x+1}{2x-5} \geq 1$$

$$\frac{x+1}{2x-5} - 1 \geq 0$$

$$\frac{x+1-1(2x-5)}{2x-5} \geq 0$$

$$\frac{-x+6}{2x-5} \geq 0$$ → zadovoljava - kada?

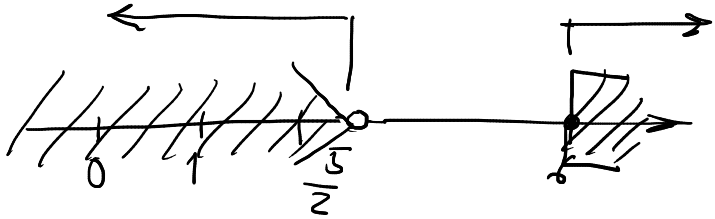
I) $-x+6 \geq 0 \Rightarrow x \leq 6$

$2x-5 > 0 \quad x > \frac{5}{2}$

$x \in \left\langle \frac{5}{2}, 6 \right]$ ✓

II) $-x+6 \leq 0 \Rightarrow x \geq 6$

$2x-5 < 0 \quad x < \frac{5}{2}$



Sustav nema rješenja!

④ (Pr 6)

$$1-\frac{2}{x-1}<\frac{2x}{x+1}$$

$$1-\frac{2}{x-1}-\frac{2x}{x+1}<0$$

$$\frac{x^2-1-2(x+1)-2x(x-1)}{(x-1)(x+1)}<0$$

$$\frac{x^2-1-2x-2-2x^2+2x}{(x-1)(x+1)}<0$$

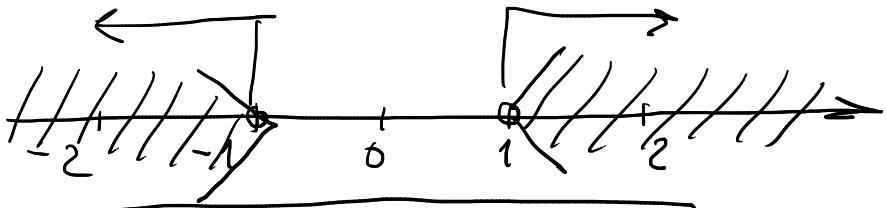
$$\frac{-x^2-3}{(x-1)(x+1)}<0$$

$-x^2-3<0$! ujk negativan!

$x^2-1>0$

$x^2>1$

$|x|>1$

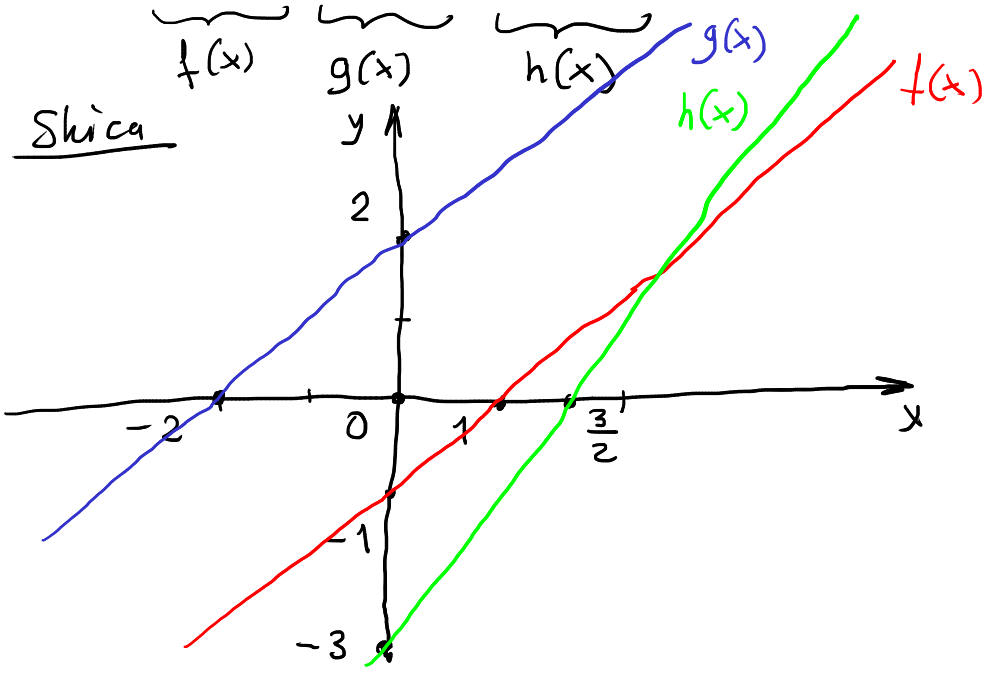


$x\in\langle-\infty,-1\rangle\cup\langle 1,+\infty\rangle$

⑤ (Pr 7 - 1)

$$\underbrace{(x-1)}_{f(x)}\underbrace{(x+2)}_{g(x)}\underbrace{(2x-3)}_{h(x)}<0$$

Skica



Proměnujeme intervaly:

$x < -2 \Rightarrow f(x), g(x), h(x) < 0$ ✓

$-2 < x < 1 \Rightarrow \left.\begin{array}{l} g(x) > 0 \\ f(x), h(x) < 0 \end{array}\right\} f(x) \cdot g(x) \cdot h(x) > 0$ (−)

$1 < x < \frac{3}{2} \quad \left.\begin{array}{l} g(x), f(x) > 0 \\ h(x) < 0 \end{array}\right\} f(x) \cdot g(x) \cdot h(x) < 0$ ✓

$x > \frac{3}{2} \quad f(x), g(x), h(x) > 0$ −

$$\boxed{x \in \langle -\infty, -2 \rangle \cup \langle 1, \tfrac{3}{2} \rangle}$$

⑥ (Př 10)

$$|3x-1| \geq \frac{1}{2}$$

1) $|3x-1| < 0$

$-(3x-1) \geq \frac{1}{2}$

$-3x \geq \frac{1}{2} - 1 = -\frac{1}{2} \quad / : (-3)$

$x \leq \frac{1}{6} \qquad x \in \langle -\infty, \frac{1}{6}]$

II) $|3x-1| > 0$

$3x - 1 \geq \frac{1}{2}$

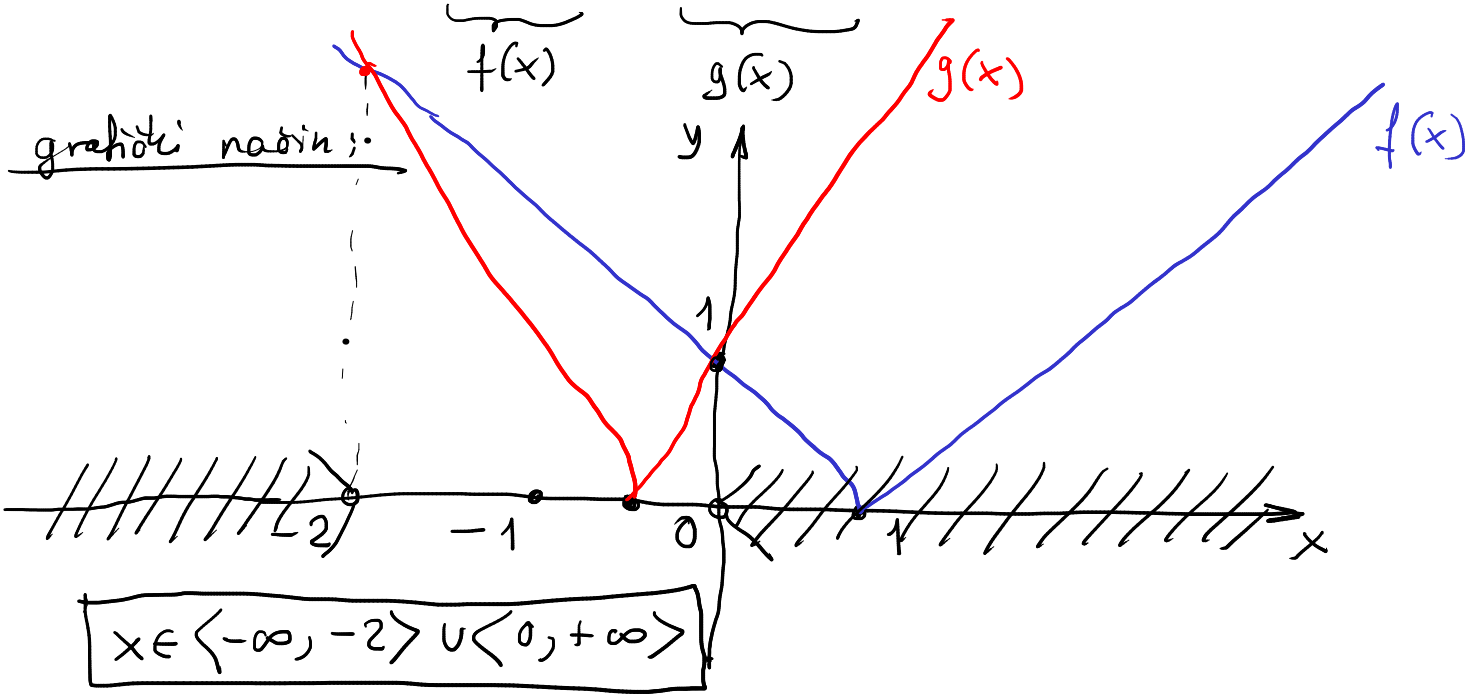
$x \geq \frac{1}{2} \qquad x \in [\frac{1}{2}, +\infty \rangle$

$$\boxed{x \in \langle -\infty, \tfrac{1}{6}] \cup [\tfrac{1}{2}, +\infty \rangle}$$

⑦ (Př 11)

$$\underbrace{|x-1|}_{f(x)} < \underbrace{|2x+1|}_{g(x)}$$

grafický nárys:



$x \in \langle -\infty, -2 \rangle \cup \langle 0, +\infty \rangle$

⑧ (zadání – 21)

$$\frac{x|x-4|-4}{1-|1-x|} \qquad 1 < x < 4 \qquad x > 0$$

I) $$\frac{-x(x-4)-4}{1-(-(1-x))} = \frac{-x^2+4x-4}{1+1-x} = \frac{-(x^2-4x+4)}{-x+2} = \frac{\cancel{(x-2)}(x-2)}{\cancel{-x+2}} = x-2$$

II) $$\frac{x(x-4)-4}{1-(1-x)} = \frac{x^2-4x-4}{1-1+x} = \boxed{\frac{x^2-4x-4}{x}}$$

⑨ (zadání 5)

$||x-1|-1|$

$x = \sqrt{2}-1$

$$||\sqrt{2}-1-1|-1| = ||\sqrt{2}-2|-1| = |-\sqrt{2}+2-1| = |-\sqrt{2}+1|$$

$$= \sqrt{2}-1$$

(10) $||x-2|-2| = ?$

$x = 1-\sqrt{3}$

$||1-\sqrt{3}-2|-2| = ||-1-\sqrt{3}|-2| = |1+\sqrt{3}-2| = |\sqrt{3}-1|$

$= \boxed{1-\sqrt{3}}$

6. SKUP KOMPLEKSNIH BROJEVA

(1) (Pr 3)

$$\left(\frac{i^{8n-5}+i^{4n+6}}{i^{12n-8}-i^{16n+9}}\right)^{44n+55} = (*)$$

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -1\cdot\underbrace{\sqrt{-1}}_{i} = -i$

$i^{8n-5} = i^{8n-8+3} = \underbrace{i^{8n-8}}_{=1}\cdot\underbrace{i^3}_{-i} = -i$

$i^{4n+6} = i^{4n+4+2} = \underbrace{i^{4n+4}}_{=1}\cdot\underbrace{i^2}_{=-1} = -1$

$i^{12n-8} = 1$

$i^{16n+9} = i^{16n+8+1} = \underbrace{i^{16n+8}}_{=1}\cdot\underbrace{i^1}_{=i} = i$

$$(*) = \left(\frac{-i-1}{1-i}\right)^{44n+55} = \left(-\frac{(1+i)}{1-i}\right)^{44n+55} = \left(-\frac{(1+i)(1+i)}{(1-i)(1+i)}\right)^{44n+55}$$

množim b. i n. sa $(1+i)$

$$= \left(\frac{-(1+2i-1)}{1-\underbrace{i^2}_{-1}}\right)^{44n+55} = \left(\frac{-2i}{2}\right)^{44n+55} = (-i)^{44n+55} = \underbrace{(-i)^{44n}}_{1}\cdot(-i)^{55}$$

$$= (-i)^{55} = \underbrace{(-i)^{52}}_{=1} \cdot \underbrace{(-i)^{3}}_{=i} = \boxed{i}$$

② (Pr 5)

$$z = \frac{1+i\sqrt{3}}{1-i} = (*)$$

$$\mathrm{Re}\,z + \mathrm{Im}\,z = ?$$

$$z = \mathrm{Re}\,z + \mathrm{Im}\,z$$

$$(*) = \frac{(1+i\sqrt{3})\cdot(1+i)}{(1-i)\cdot(1+i)} = \frac{1+i-\sqrt{3}+i\sqrt{3}}{\underbrace{1-i^2}_{=2}}$$

$$= \frac{1-\sqrt{3}+i(1+\sqrt{3})}{2} = \underbrace{\frac{1-\sqrt{3}}{2}}_{\mathrm{Re}\,z} + \underbrace{\frac{1+\sqrt{3}}{2}}_{\mathrm{Im}\,z} i$$

$$\boxed{\mathrm{Re}\,z + \mathrm{Im}\,z = \frac{1-\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2} = \frac{1}{2} + \frac{1}{2} = 1}$$

③ (Pr 6)

$$\frac{a+bi}{1-i} = \frac{5}{1-2i} \quad / \cdot (1-i)\cdot(1-2i)$$

$$(a+bi)(1-2i) = 5(1-i)$$

$$a - 2ai + bi + 2b = 5 - 5i$$

$$a+2b+(-2a+b)i = 5-5i$$

$$\left.\begin{array}{l} a+2b = 5 \\ -2a+b = -5 \quad / \cdot(-2) \end{array}\right\} (+)$$

$$\left.\begin{array}{l} a+2b = 5 \\ 4a-2b = 10 \end{array}\right\} (+)$$

$$5a = 15 \Rightarrow \boxed{a = 3}$$

$$\boxed{b} = -5 + 2a = -5 + 2\cdot 3 = \boxed{1}$$

④ (Pr 8)

$$\left| \frac{(1-2i)^5}{(2-i)^7 \cdot (1-i)^4} \right| =$$

$$= \frac{|(1-2i)^5|}{|(2-i)^7 \cdot (1-i)^4|} = \frac{|(1-2i)^5|}{|(2-i)^7| \cdot |(1-i)^4|}$$

$$= \frac{|1-2i|^5}{|2-i|^7 \cdot |1-i|^4} = (\bullet)$$

$$|1-2i| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

$$|2-i| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$|1-i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$(\bullet) = \frac{(\sqrt{5})^5 \cdot 1}{(\sqrt{5})^7 \cdot (\sqrt{2})^4} = \frac{1}{5 \cdot 4} = \boxed{\frac{1}{20}}$$

⑤ (Pr 9)

$$z^2 + \bar{z}^2 = 6$$

$$z \cdot \bar{z} = 5$$

$$\begin{cases} z = a + bi \\ \bar{z} = a - bi \end{cases}$$

$$(a+bi)^2 + (a-bi)^2 = 6$$

$$a^2 + 2abi - b^2 + a^2 - 2abi - b^2 = 6$$

$$2a^2 - 2b^2 = 6 \quad / : 2$$

$$a^2 - b^2 = 3$$

$$(a+bi)(a-bi) = 5$$

$$a^2 - b^2 \cdot (-1) = 5$$

$$a^2 + b^2 = 5$$

(+)

$$2a^2 = 8 \quad / : 2$$

$$a^2 = 4$$

$$a = \pm 2 \qquad b = \pm 1$$

$$z_{1,2} = \pm(2+i)$$
$$z_{3,4} = \pm(2-i)$$

⑥ (Pr 12)

$$|z-2|^2 + |z+2|^2 = 26$$

$$z = x + yi$$

$$|x-2+yi|^2 + |x+2+yi|^2 = 26$$

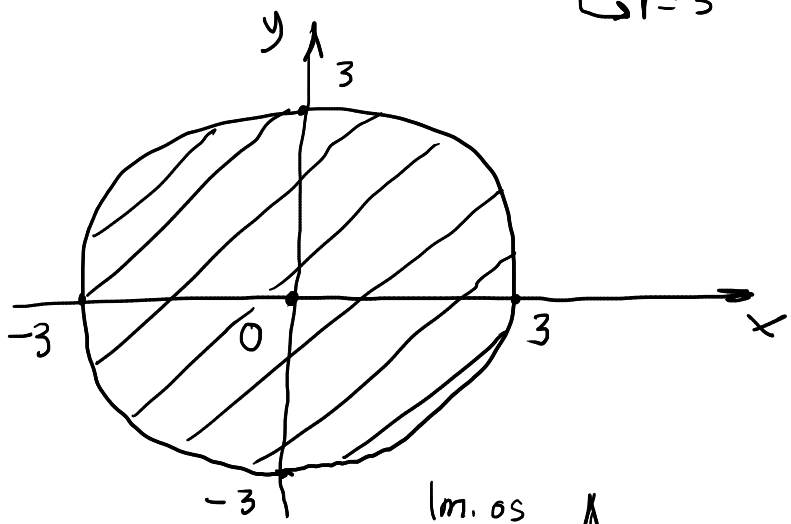
$$(x-2)^2 + y^2 + (x+2)^2 + y^2 = 26$$

$$x^2 - 2\cdot 2x + 4 + y^2 + x^2 + 4x + 4 + y^2 = 26$$

$$2x^2 + 2y^2 = 26 - 4 - 4 = 18 \;/:2$$

$$\boxed{x^2 + y^2 = 9} \rightarrow \text{jednadžba kružnice}$$

$r = 3$

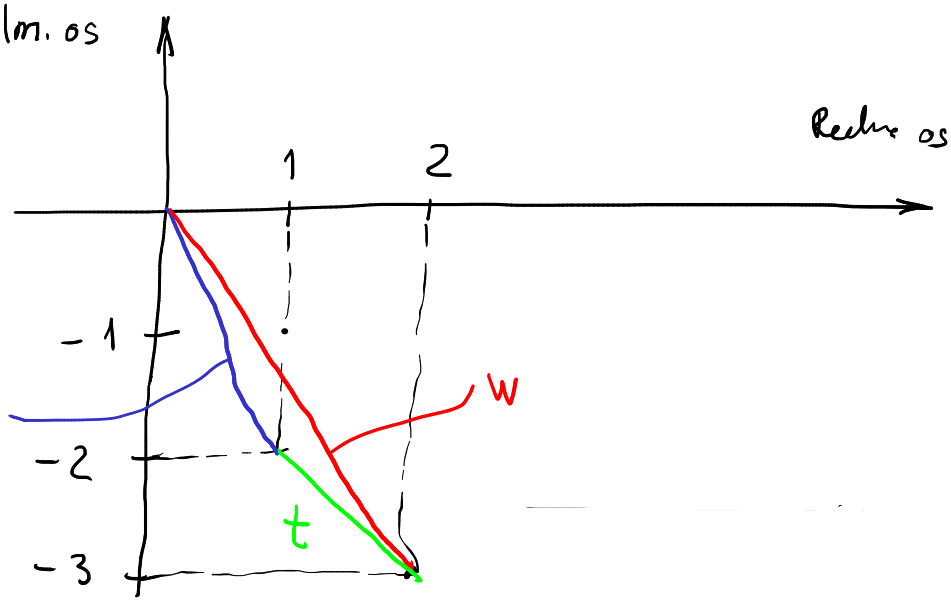


⑦ (zadaci -18)

$z = 1 - 2i$

$w = 2 - 3i$

$t = ?$



$$t = \sqrt{(2-1)^2 + (-3-(-2))^2}$$

$$t = \sqrt{1^2 + (-1)^2}$$

$$\boxed{t = \sqrt{2}}$$

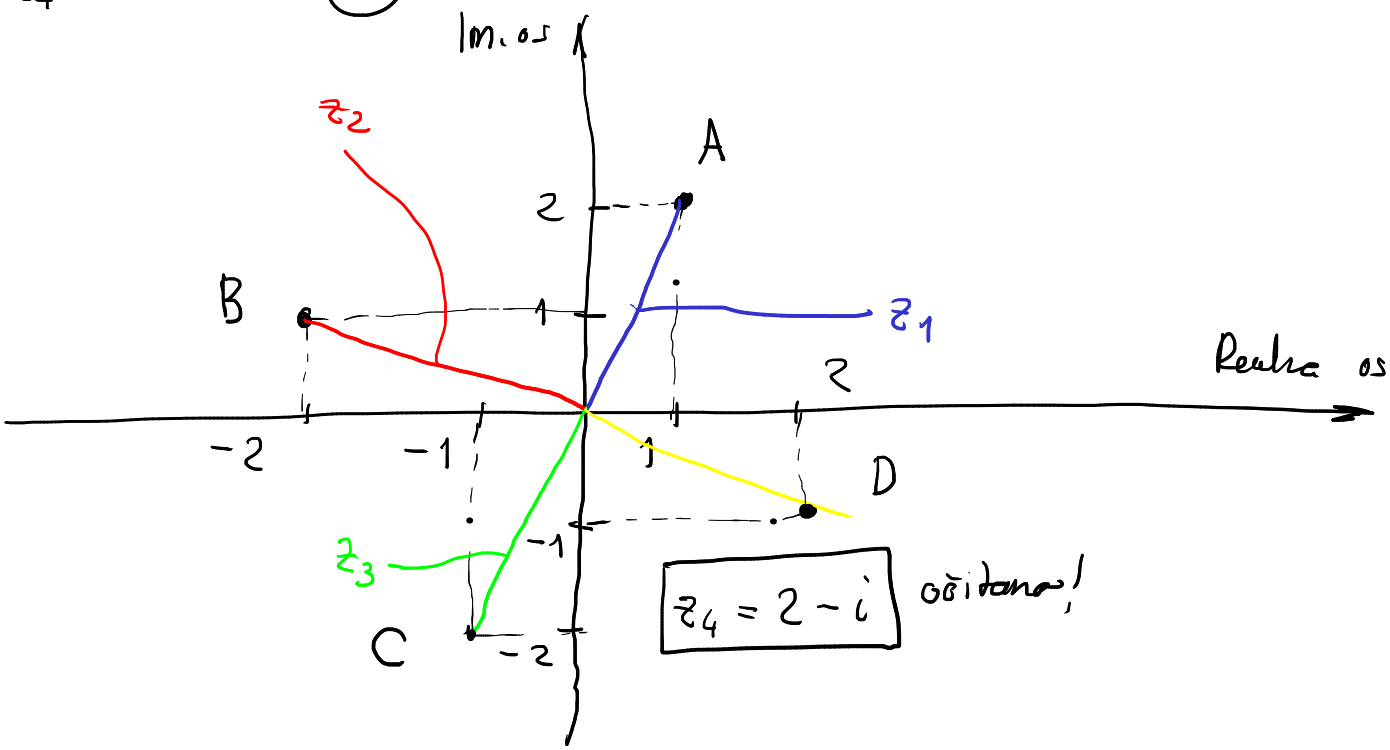
(8) (zadaci - 19)

$z_1 = 1 + 2i$ (A)

$z_2 = -2 + i$ (B)

$z_3 = -1 - 2i$ (C)

$z_4 = ?$ (D)



(9) $\frac{(1+ai)(1-2i) = (1+i)(1-bi)}{a+b = ?}$

$1 - 2i + ai + 2a = 1 - bi + i + b$

$(1+2a) + (a-2)i = (1+b) + (1-b)i$

$1 + 2a = 1 + b$

$a - 2 = 1 - b$ } (+)

$-1 + 3a = 2$

$3a = 2 + 1 = 3 \Rightarrow a = 1$

$1 + 2 = 1 + b \Rightarrow b = 2$

$\boxed{a+b = 1+2 = 3}$