16. KOORDINATNI SUSTAV U RAVNINI

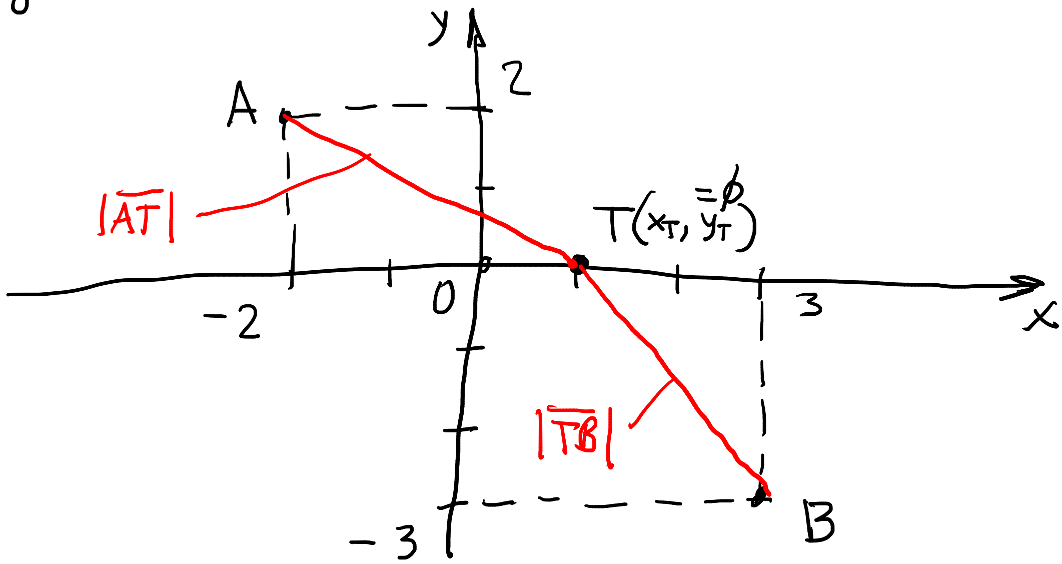
1 (Pr 3)

$A(-2,2)$
$B(3,-3)$

$T(x_T, \underbrace{y_T}_{=0})$
↳ leži na osi x – apcisi

$|\overline{AT}| = |\overline{TB}|$

– udaljenost točaka u ravnini



$$\sqrt{(y_T-y_A)^2+(x_T-x_A)^2} = \sqrt{(y_T-y_B)^2+(x_T-y_B)^2}$$

$$\sqrt{(\underbrace{y_T}_{=0}-2)^2+(x_T-(-2))^2} = \sqrt{(0-(-3))^2+(x_T-3)^2} \Big/^2$$

$$4+(x_T+2)^2 = 9+(x_T-3)^2$$

$$4+\cancel{x_T^2}+4x_T+4 = 9+\cancel{x_T^2}-6x_T+9$$

$$10x_T = 18-8=10$$

$$\Rightarrow \boxed{x_T=1} \qquad \boxed{T(1,0)}$$

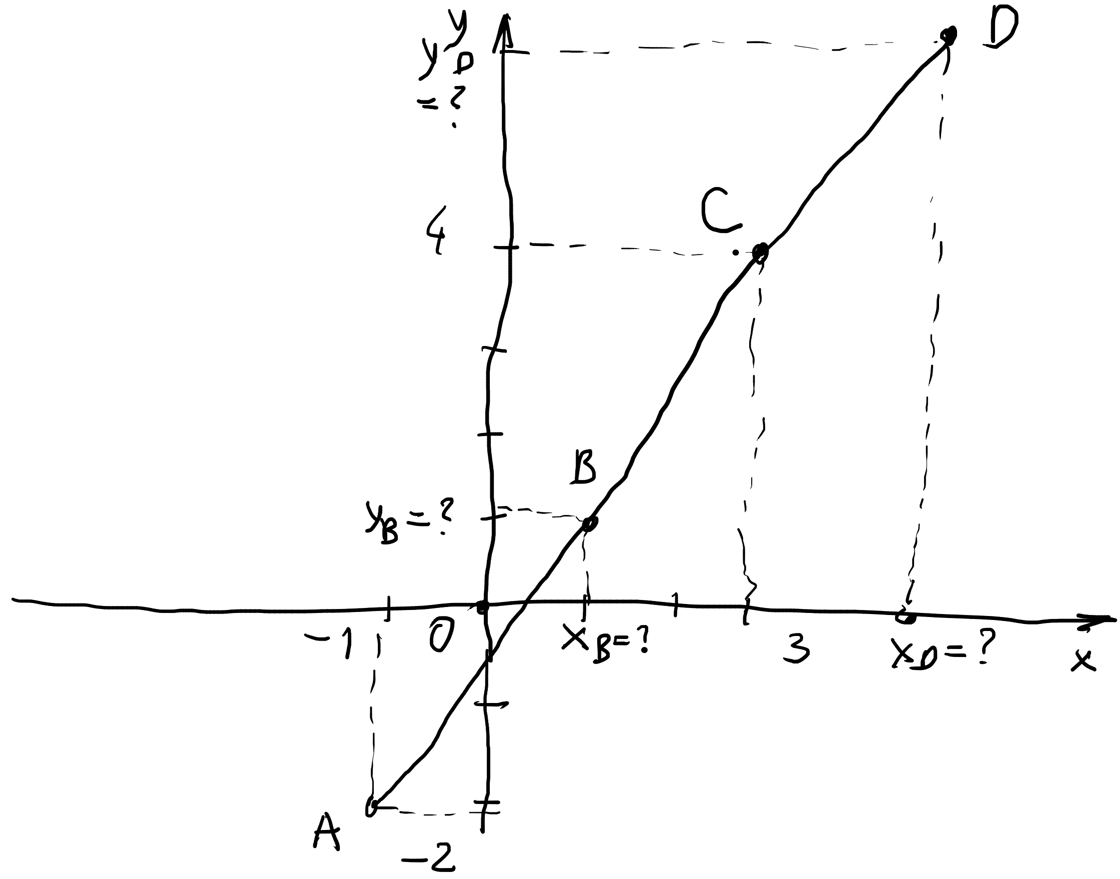
2 (Pr 4)

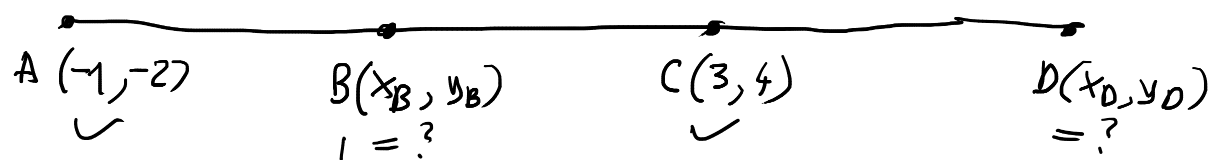
sukladni dijelovi – jednaki po duljini

$A(-1,-2)$

$C(3,4)$

$B(x_B, y_B)=?$ $D(x_D, y_D)=?$





↳ leži u polovištu dužine $\overline{AC}$

$$\boxed{x_B} = \frac{x_A + x_C}{2} = \frac{-1+3}{2} = \frac{2}{2} = \boxed{1}$$

$$\boxed{y_B} = \frac{y_A + y_C}{2} = \frac{-2+4}{2} = \frac{2}{2} = \boxed{1}$$

$$\boxed{B(1,1)}$$

C – polovište dužine $\overline{BD}$

$$x_C = \frac{x_B + x_D}{2} \Rightarrow \boxed{x_D} = 2x_C - x_B = 2 \cdot 3 - 1 = 6 - 1 = \boxed{5}$$

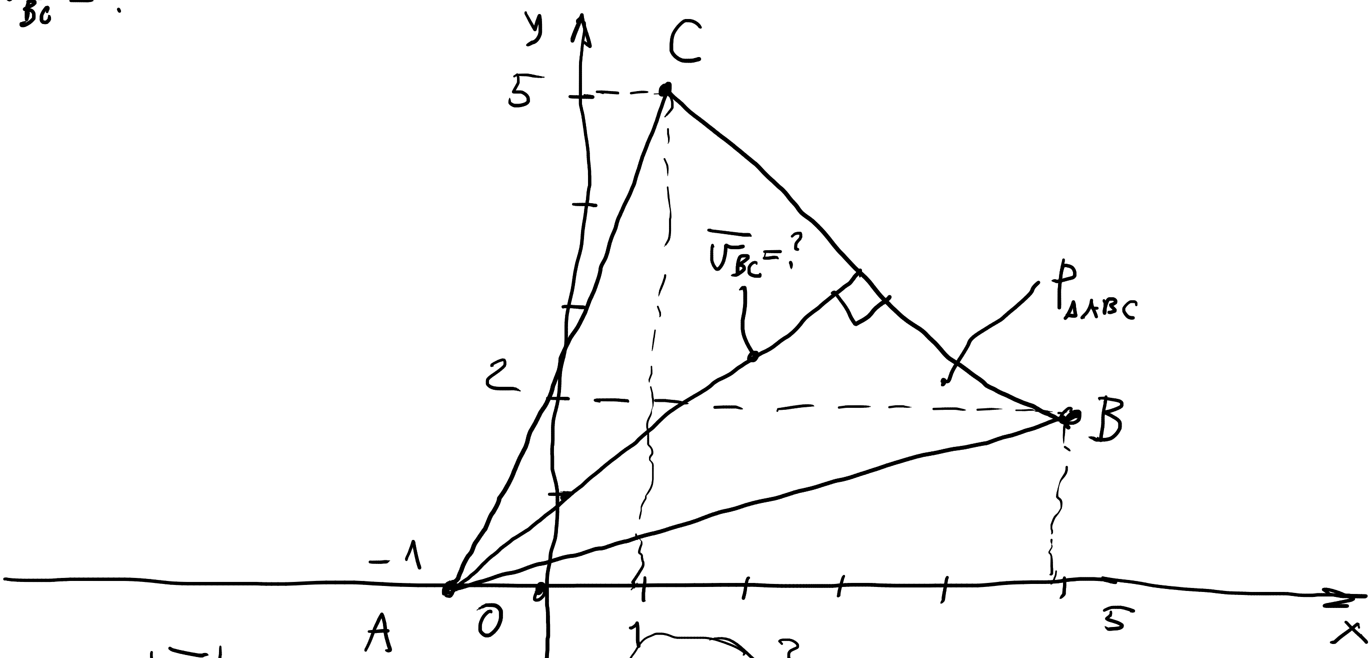
$$y_C = \frac{y_B + y_D}{2} \Rightarrow \boxed{y_D} = 2y_C - y_B = 2 \cdot 4 - 1 = 8 - 1 = \boxed{7}$$

$$\boxed{D(5,7)}$$

③ (Pr 7)

$$\left.\begin{array}{l} A(-1,0) \\ B(5,2) \\ C(1,5) \end{array}\right\} \text{vrhovi trokuta } ABC$$

$v_{\overline{BC}} = ?$



$$P_{\Delta ABC} = \frac{|\overline{BC}| \cdot v_{\overline{BC}}}{2} \Rightarrow v_{\overline{BC}} = \frac{2 \cdot P_{\Delta ABC} \, ?}{|\overline{BC}| \, ?}$$

$$P_{\Delta ABC} = \frac{1}{2}\left[x_A(y_B - y_C) + x_B(y_C - y_A) + x_C(y_A - y_B)\right]$$

$$P_{\Delta ABC} = \frac{1}{2}\left[-1(2-5) + 5(5-0) + 1(0-2)\right] = 13$$

$$|\overline{BC}| = \sqrt{(y_C - y_B)^2 + (x_C - x_B)^2} = \sqrt{(5-2)^2 + (1-5)^2} = \sqrt{9+16} = 5$$

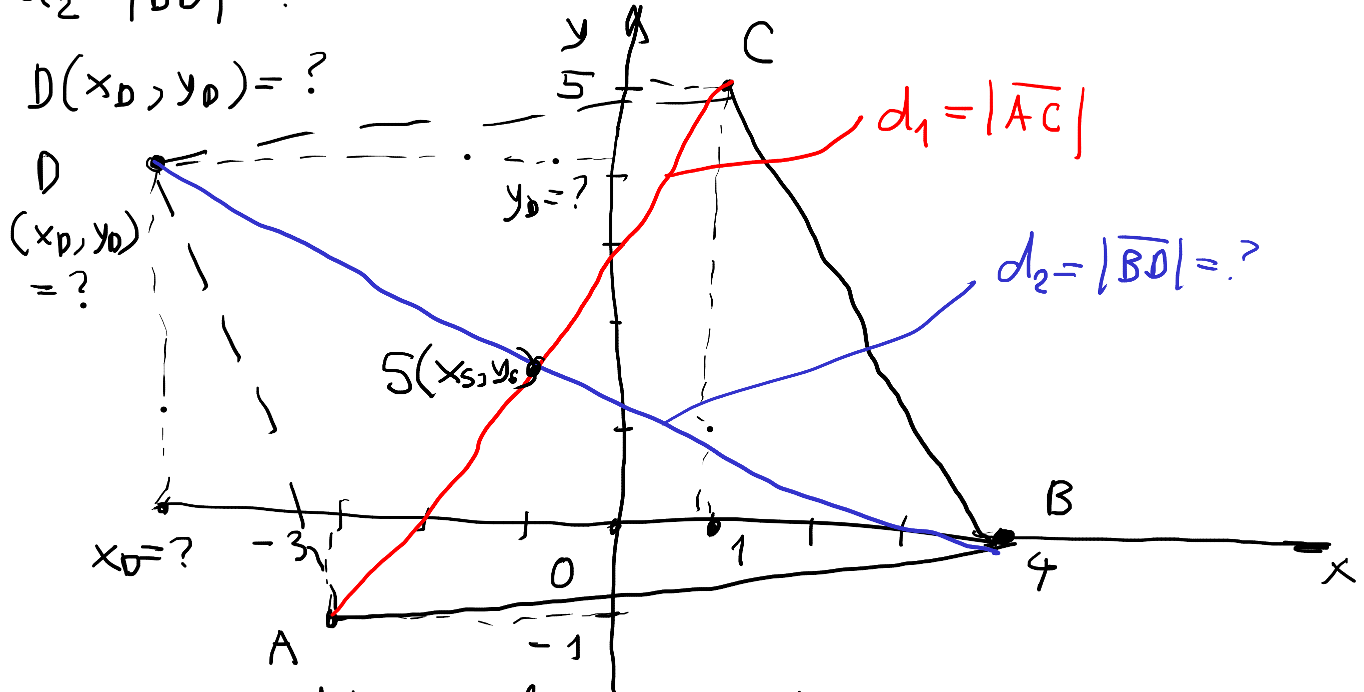
$$\boxed{v_{\overline{BC}} = \frac{2 \cdot 13}{5} = \frac{26}{5} = 5{,}2}$$

④ (Pr 5)

$A(-3,-1)$
$B(4,6)$
$C(1,5)$
} tri nastupna vrha paralelograma ABCD

$d_2 = |\overline{BD}| = ?$

$D(x_D, y_D) = ?$



S - sjecište dijagonala d_1 i d_2
- polovište dužine (dijagonale) $d_1 = |\overline{AC}|$

$$x_S = \frac{x_A + x_C}{2} = \frac{-3+1}{2} = \frac{-2}{2} = -1$$

$$y_S = \frac{y_A + y_C}{2} = \frac{-1+5}{2} = \frac{4}{2} = 2$$

$S(-1, 2)$

polovište dijagonale $d_2 = |\overline{BD}|$

$$x_S = \frac{x_B + x_D}{2} \Rightarrow \boxed{x_D = 2x_S - x_B = 2\cdot(-1) - 4 = -2-4 = -6}$$

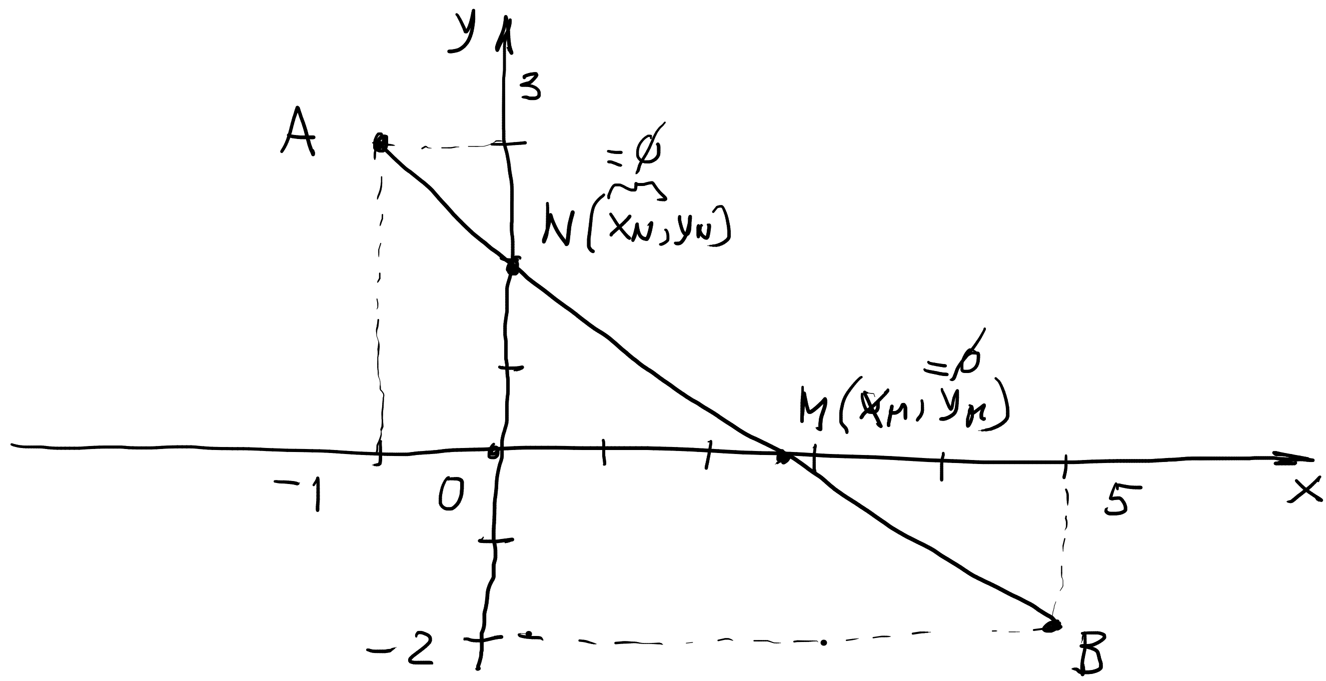
$$y_S = \frac{y_B + y_D}{2} \Rightarrow \boxed{y_D = 2y_S - y_B = 2\cdot 2 - 0 = 4 - 0 = 4}$$

$$\boxed{D(-6, 4)}$$

5 (Pr 9)

$$\left.\begin{array}{l} A(-1,3) \\ B(5,-2) \end{array}\right\} \overline{AB}$$

$$\left.\begin{array}{l} \lambda_1 = ? \\ \lambda_2 = ? \end{array}\right\}$$ omjeri sjecišta s koordinatnim osima i dužine dužine $\overline{AB}$



$$\lambda_1 = \frac{|\overline{AM}|}{|\overline{MB}|}$$

$$\lambda_2 = \frac{|\overline{AN}|}{|\overline{NB}|}$$

$$y_M = \frac{y_A + y_B \cdot \lambda_1}{1 + \lambda_1}$$

$$0 = \frac{3 + (-2) \cdot \lambda_1}{1 + \lambda_1} \Rightarrow \boxed{\lambda_1 = \frac{3}{2}}$$

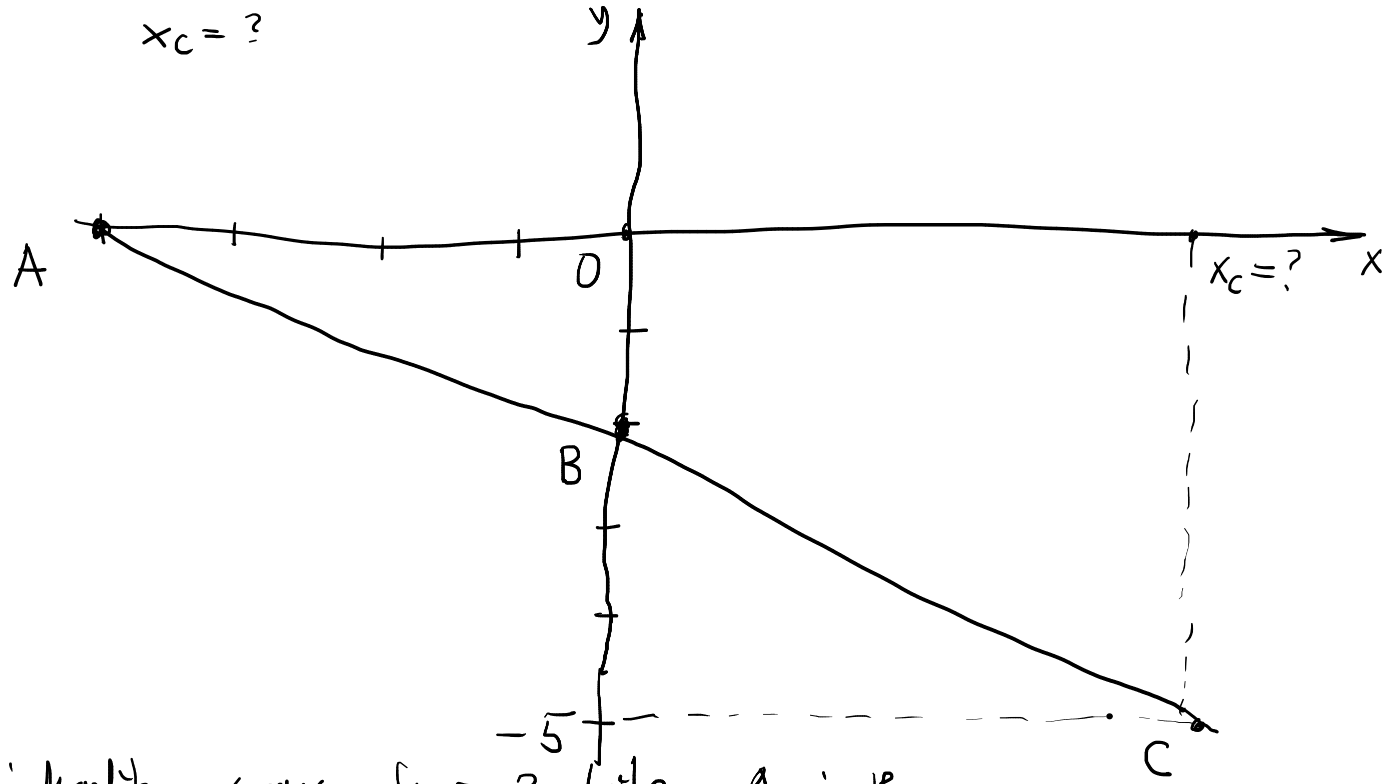
$$x_N = \frac{x_A + x_B \cdot \lambda_2}{1 + \lambda_2}$$

$$0 = \frac{-1 + 5 \cdot \lambda_2}{1 + \lambda_2} \Rightarrow \boxed{\lambda_2 = \frac{1}{5}}$$

⑥ (zadaci - 13)

$A(-4,0)$
$B(0,-2)$
$C(x_0, -5)$
} A, B, C - leže na istom pravcu!

$x_C = ?$



jednadžba pravca kroz 2 točke - A i B

$$y - y_A = \frac{y_B - y_A}{x_B - x_A}(x - x_A)$$

$$y - 0 = \frac{-2-0}{0-(-4)}(x-(-4))$$

$$y = -\frac{2}{4}(x+4) = -\frac{1}{2}(x+4) = -\frac{1}{2}x - 2$$

točka C - koordinata x:

$$y = 5 = -\frac{1}{2}x - 2$$

$$-\frac{1}{2}x = -3 \;/\cdot(-2)$$

$$\boxed{x_C = 6}$$

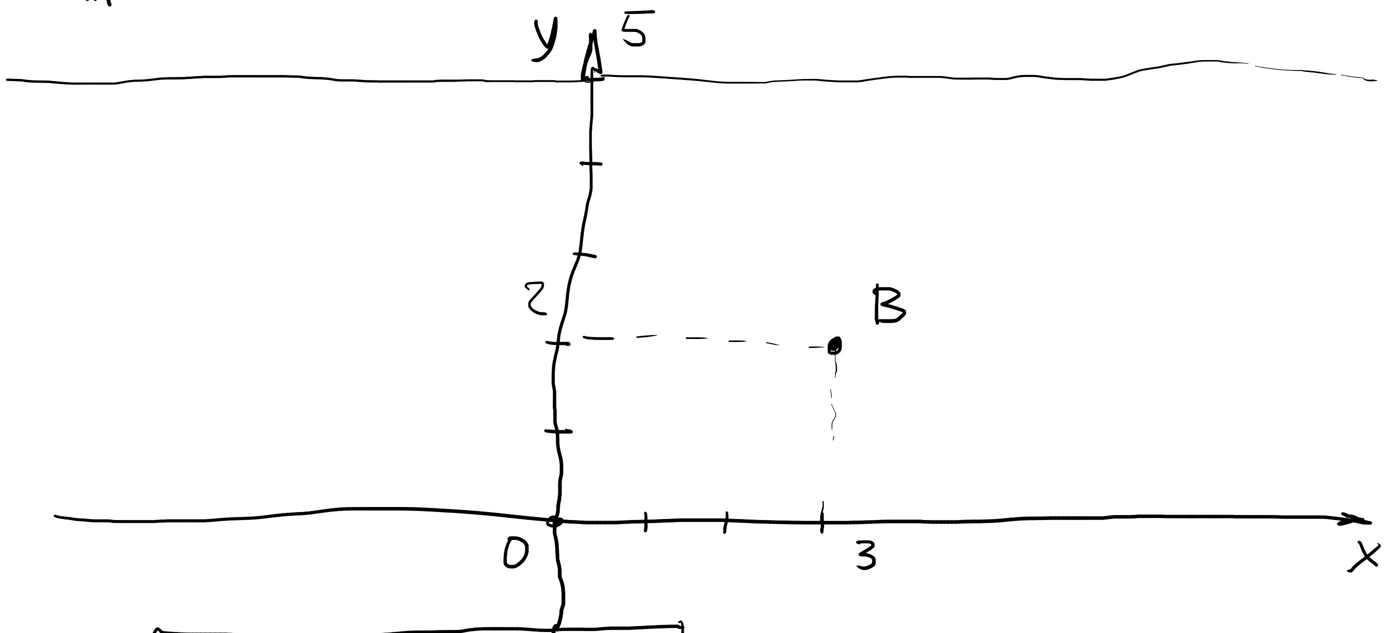
⑦ (zadaci - 4)

$A(\underbrace{x_A}_{=?}, 5)$

$|Ax| = |\overline{AB}|$

$B(3,2)$

$x_A = ?$



$$\Leftrightarrow \begin{cases} |\overline{AB}| = \sqrt{(y_B - y_A)^2 + (x_B - x_A)^2} = \sqrt{(2-5)^2 + (3-x_A)^2} \\ |\underbrace{Ax}_{x_A}| = 5 \end{cases}$$

$$\to \sqrt{9 + (3 - x_A)^2} = 5 \,/^2 \implies 9 + (3 - x_A)^2 = 25$$

$$(3 - x_A)^2 = 25 - 9 = 16 \,/\sqrt{\ }$$

2 rješenja:

I) $3 - x_A = 4$
$\to \boxed{(x_A)_1 = -1}$

II) $-(3 - x_A) = 4$
$\boxed{(x_A)_2 = 7}$

17. PRAVAC

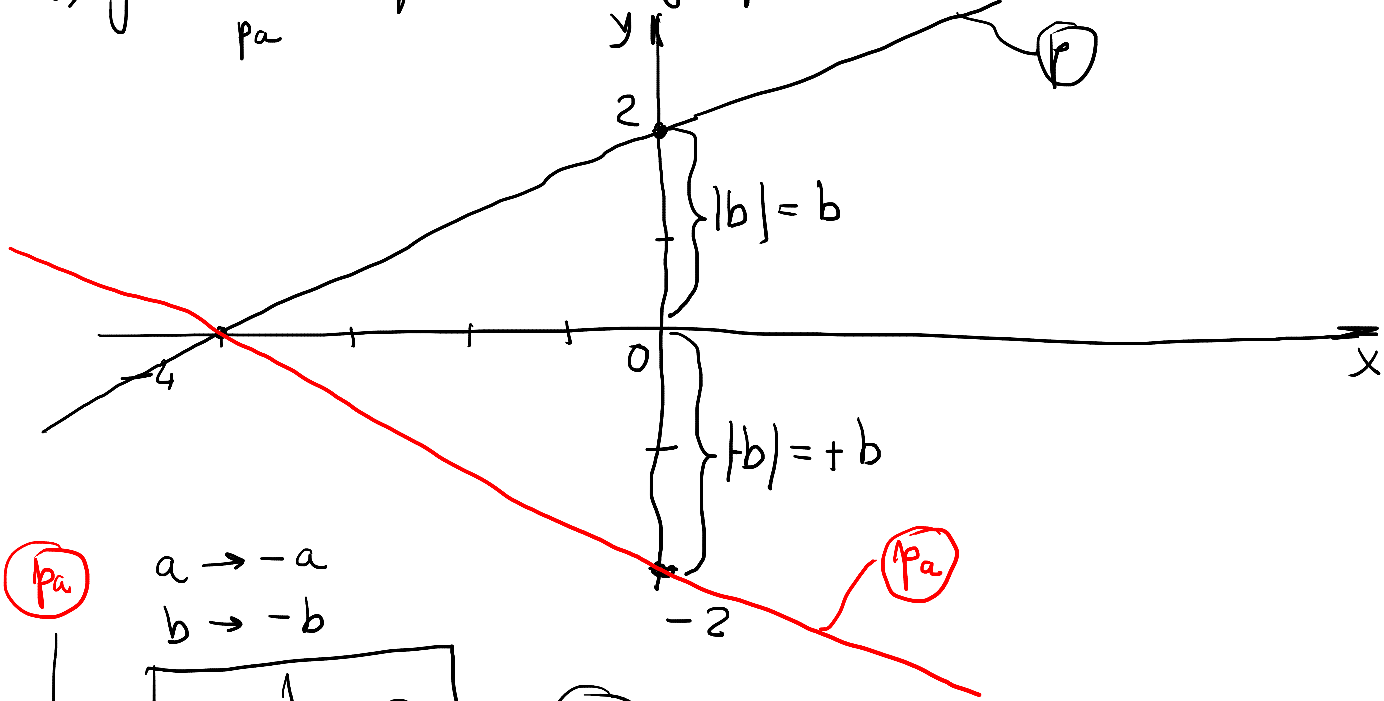
① (pr 1)

$p \ldots \; x - 2y + 4 = 0 \implies -2y = -4 - x \; / : (-2)$

implicitni oblik

$y = \frac{1}{2}x + 2$ eksplicitni oblik

a) jednadžba pravca koja je p simetričan s obzirom na os x: pa



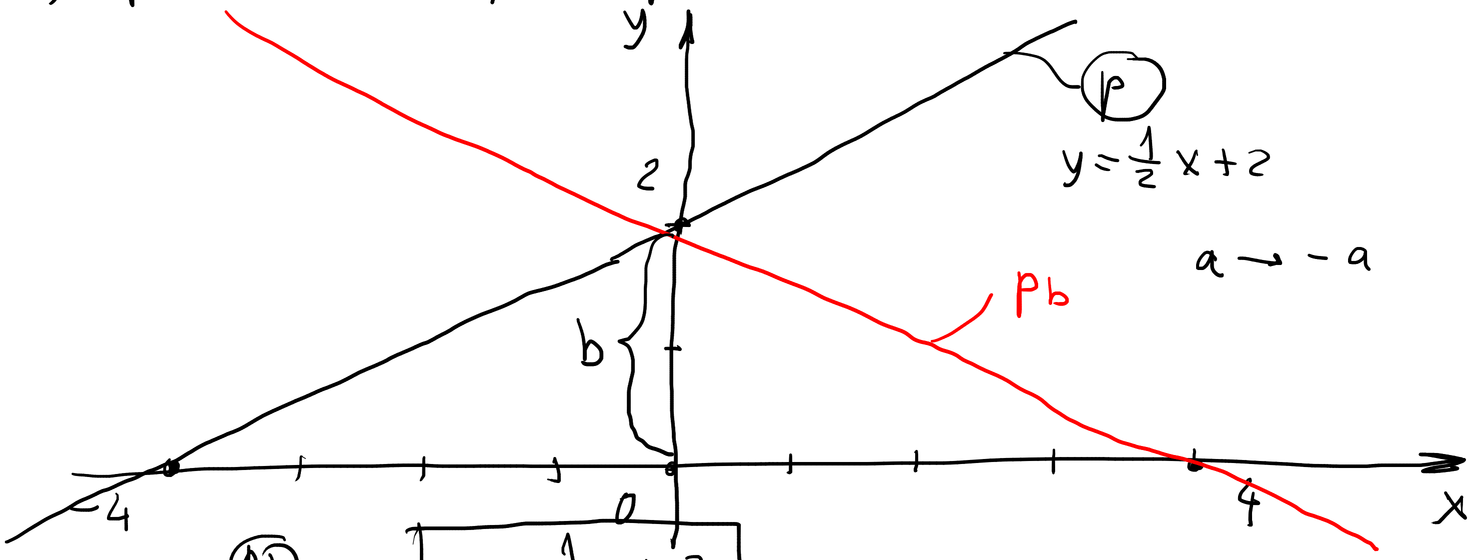
pa

$a \to -a$

$b \to -b$

$\boxed{y = -\frac{1}{2}x - 2} \ldots$ pa

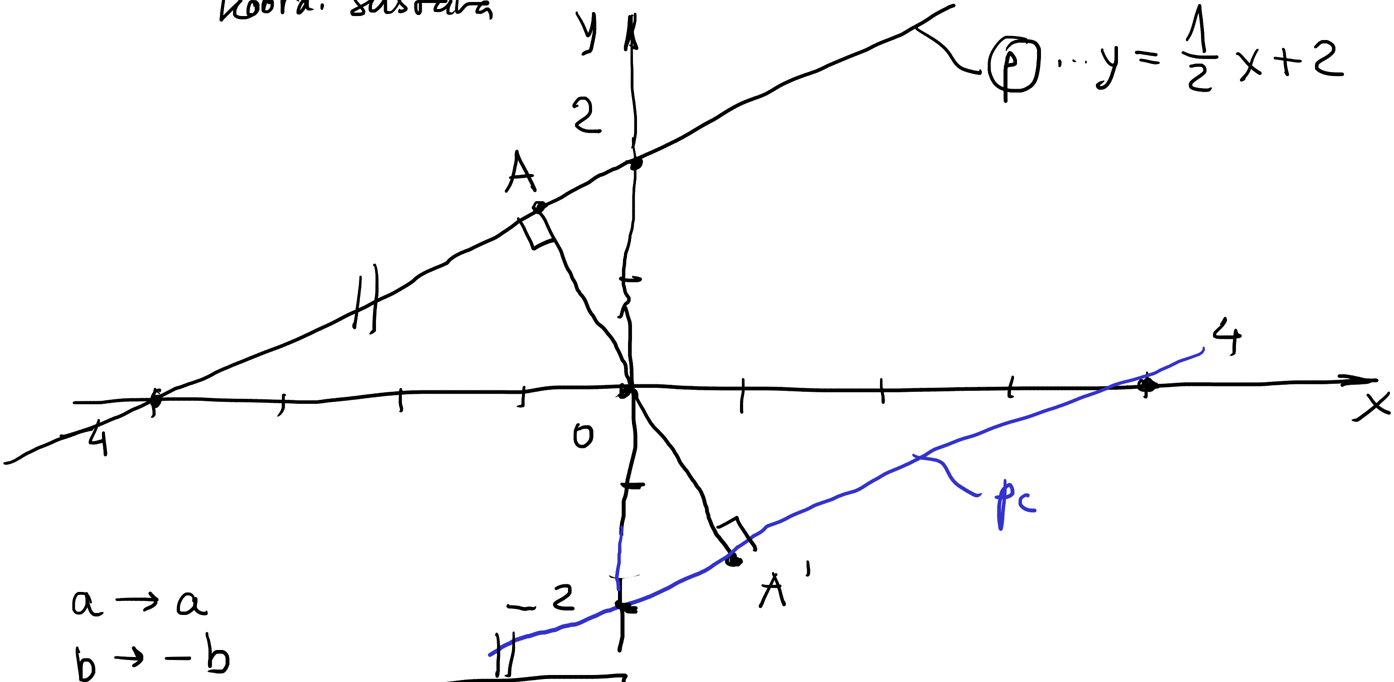
b) pb – simetričan pravcu p s obzirom na y os



$a \to -a$

pb ... $\boxed{y = -\frac{1}{2}x + 2}$

c) p_c – simetričan pravac p s obzirom na ishodište koord. sustava



$a \to a$
$b \to -b$

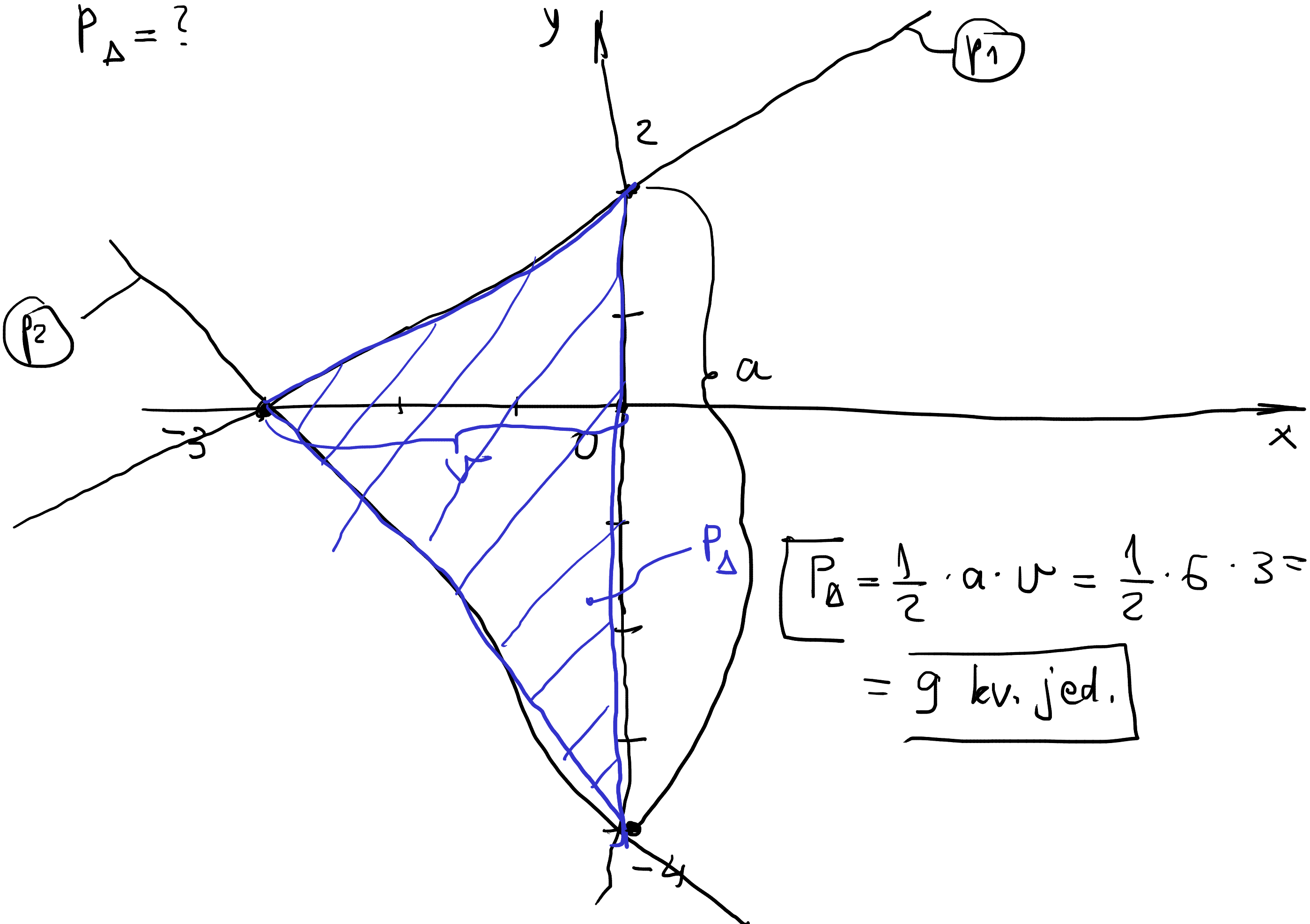
$p_c \ldots \boxed{y = \frac{1}{2}x - 2}$

2 (Pr 4)

$p_1 \ldots 2x - 3y + 6 = 0 \Rightarrow y = \frac{2}{3}x + 2$

$p_2 \ldots 4x + 3y + 12 = 0 \Rightarrow y = -\frac{4}{3}x - 4$

$P_\Delta = ?$



$\boxed{P_\Delta = \frac{1}{2} \cdot a \cdot v = \frac{1}{2} \cdot 6 \cdot 3 = \frac{18}{2}}$

$= \boxed{9 \text{ kv. jed.}}$

③ (Pr 3)

p_1 ... prolazi tačkom $T(2,1)$

$\varphi_1 = 2\varphi_2$

p_2 ..... $2x - 3y + 5 = 0 \Rightarrow y = \underbrace{\frac{2}{3}}_{a_2}x + \frac{5}{3}$

$a_2 = tg\,\varphi_2 = \frac{2}{3}$

$\Rightarrow \varphi_2 = arctg\left(\frac{2}{3}\right) = 33{,}69^\circ$

$\varphi_1 = 2\varphi_2 = 2 \cdot 33{,}69^\circ = 67{,}38^\circ$

$tg\,\varphi_1 = tg(67{,}38^\circ) \approx 2{,}4 = a_1$

p_1 ... $y = a_1x + b_1 = 2{,}4x + b_1$?

$T_1 \in p_1$

$y(x=2) = 1 = 2{,}4 \cdot 2 + b_1 \Rightarrow b_1 = -3{,}8$

p_1 ... $\boxed{y = 2{,}4x - 3{,}8}$

④ $T(-1,4)$ – prolaze 2 okomita pravca p_1 i p_2

$P_\Delta = 20$ kv. jed. sa osi x

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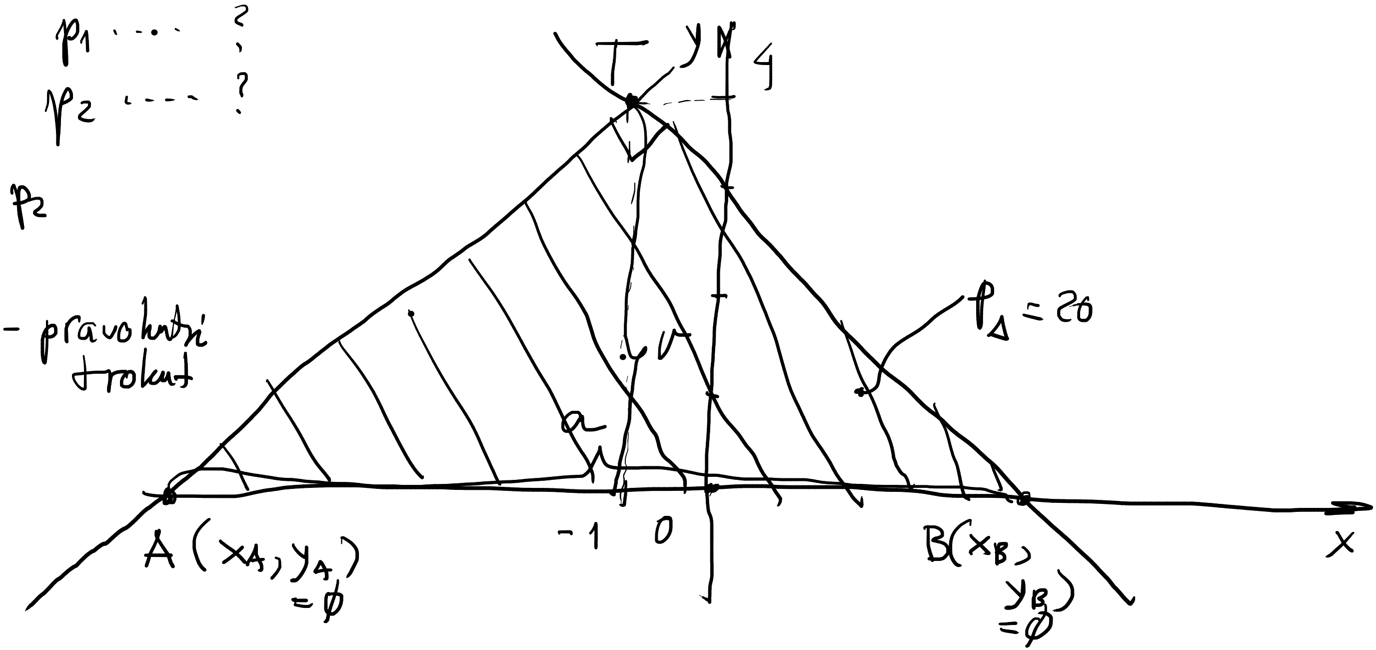
p_1 .... ?

p_2 .... ?

$p_1 \perp p_2$

ΔABT – pravougli trokut



$P_\Delta = \frac{1}{2} a v \quad (1)$

$v = 4 = y_T$

$a = |x_B - x_A| \quad (2)$

$\left.\begin{array}{l} p_1 \dots k_1 \\ p_2 \dots k_2 \end{array}\right\} p_1 \perp p_2 \Rightarrow k_2 = -\frac{1}{k_1}$

$T \in p_1, p_2 \quad \left.\begin{array}{l} x_T = -1 \\ y_T = 4 \end{array}\right\}$ $p_1 \dots y = k_1 x + b_1$

$4 = (-1) \cdot k_1 + b_1$

$\Rightarrow b_1 = 4 + k_1$

$p_2 \quad 4 = (-1) \cdot \underbrace{k_2}_{= -\frac{1}{k_1}} + b_2 = \frac{1}{k_1} + b_2 \Rightarrow b_2 = 4 - \frac{1}{k_1}$

$p_1 \dots y = k_1 x + b_1 = k_1 x + k_1 + 4$

$p_2 \dots y = k_2 x + b_2 = -\frac{1}{k_1} x + 4 - \frac{1}{k_1}$

$|x_B - x_A| = ?$

odsječci pravaca p_1 i p_2 sa osi x:

$p_1 \quad y = 0 \quad 0 = k_1 x_1 + k_1 + 4 \Rightarrow x_1 = -\frac{k_1 + 4}{k_1} = x_A$

$p_2 \quad y = 0 \quad 0 = -\frac{1}{k_1} x_2 - \frac{1}{k_1} + 4 \Rightarrow x_2 = -1 + 4k_1 = x_B$

$|x_B - x_A| = \left| -1 + 4k_1 - \left(-\frac{(k_1 + 4)}{k_1} \right) \right| = \left| \frac{4k_1^2 + 4}{k_1} \right| \quad (3) \rightarrow (1)$

$P_\Delta = \frac{1}{2} a v$

$20 = \frac{1}{\cancel{2}_1} \cdot \left| \frac{4k_1^2 + 4}{k_1} \right| \cdot \cancel{4}^{2} \quad / : 2$

$$\left|\frac{4k_1^2+4}{k_1}\right| = 10 \quad / \cdot |k_1|$$

$$\underbrace{|4k_1^2+4|}_{>0} = 10|k_1|$$

$$4k_1^2+4-10|k_1| = 0 \quad / : 2$$

$$2k_1^2 - 5\underbrace{|k_1|}+2 = 0$$

→ imamo 2 slučaja: I) $k_1 > 0$
II) $k_1 < 0$

I) $2k_1^2 - 5k_1 + 2 = 0$

→ $(k_1)_1 = \frac{1}{2}$

→ $(k_1)_2 = 2$

II) $2k_1^2 + 5k_1 + 2 = 0$

→ $(k_1)_3 = -2$

→ $(k_1)_4 = -\frac{1}{2}$

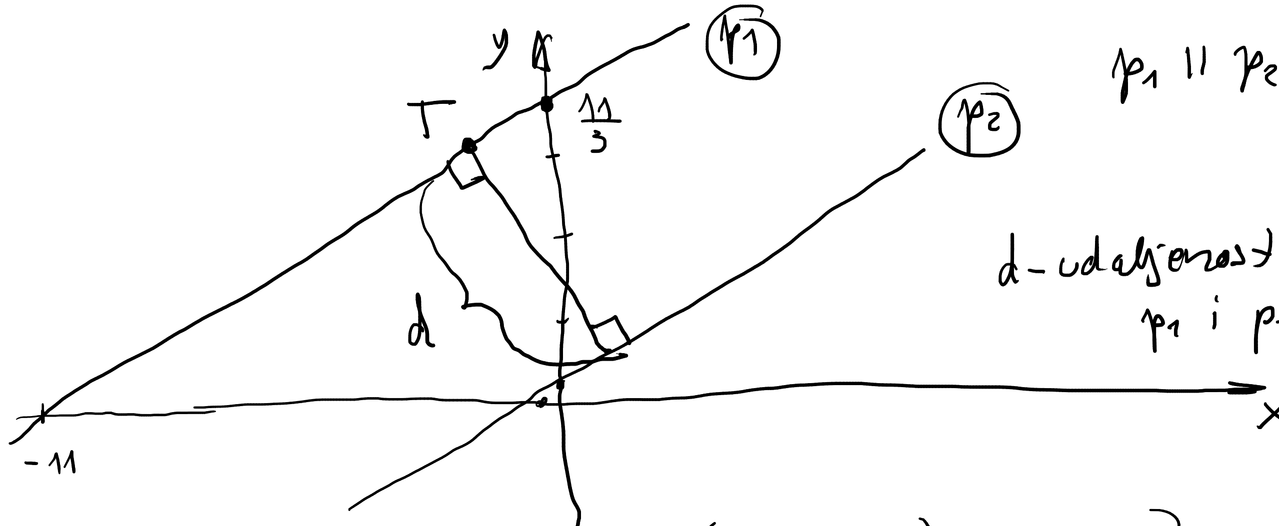
I)	II)
$(p_1)_I \quad y = 2x+6$	$(p_1)_{II} \quad y = -\frac{1}{2}x + \frac{9}{2}$
$(p_2)_I \quad y = -\frac{1}{2}x + \frac{7}{2}$	$(p_2)_{II} \quad y = -2x+2$

5 (Pr 9)

$p_1 \cdots 4x - 3y + 11 = 0 \Rightarrow y = \frac{4}{3}x + \frac{11}{3}$ (1)

$p_2 \cdots 4x - 3y + 1 = 0 \Rightarrow y = \frac{4}{3}x + \frac{1}{3}$ (2)

$P_{\square} = ?$



$p_1 \parallel p_2$

d - udaljenost pravaca p_1 i p_2 - stranica kvadrata

uzmemo po volji točku $T(x_T = 1, y_T) \to p_1$

$$y(x=1) = \frac{4}{3} \cdot 1 + \frac{11}{3} = \frac{15}{3} = 5$$

$T(1, 5)$

$$d(T, p_2) = \frac{|Ax_T + By_T + C|}{\sqrt{A^2 + B^2}} = \frac{|4 - 15 + 1|}{\sqrt{4^2 + (-3)^2}} = \frac{10}{5} = 2$$

$$\boxed{P = d^2 = 2^2 = 4 \text{ kv. jed.}}$$

6 (Pr 10)

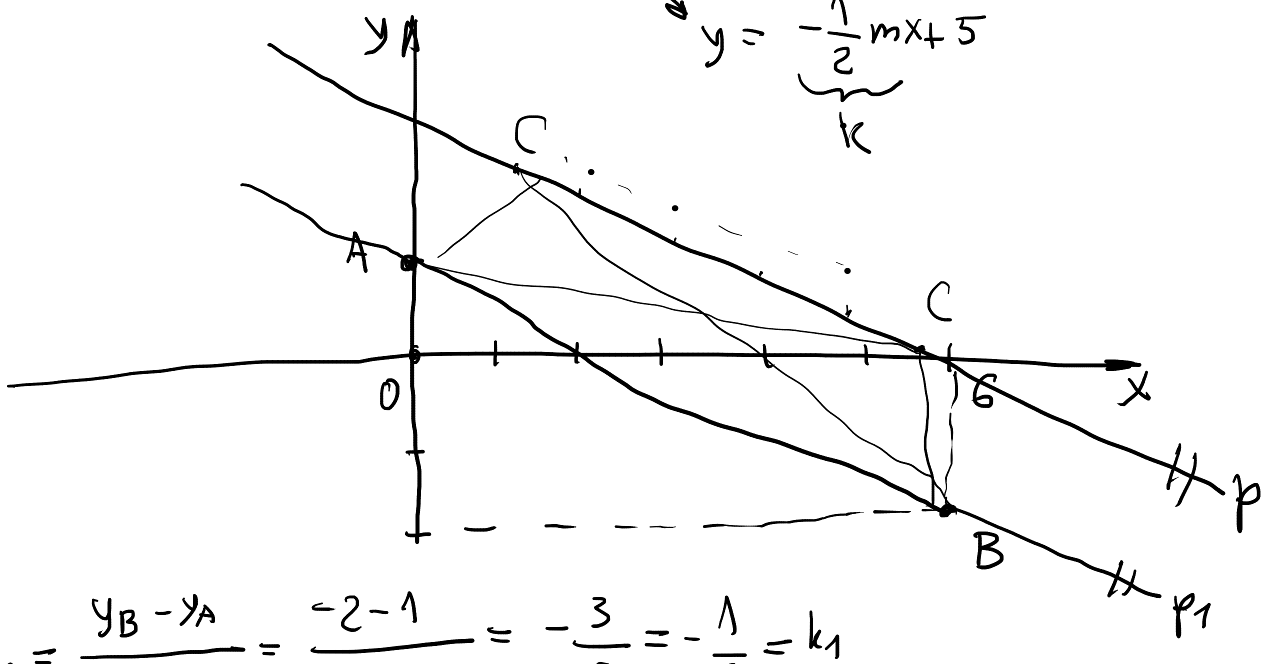
A(0, 1)
B(6, -2) } vrhovi trokuta ABC

p_1 ... A, B

$p_1 \parallel p$ $k = k_1$

C ... p ... $mx + 2y - 10 = 0$

$$y = \underbrace{-\frac{1}{2}m}_{k}x + 5$$



$$k_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{-2-1}{6-0} = -\frac{3}{6} = -\frac{1}{2} = k_1$$

$$k = k_1 = -\frac{1}{2} = -\frac{1}{2}m \Rightarrow \boxed{m = 1}$$

(7) sustav nejednadžbi

(p1) ... $x-y+2\geq 0$

$y\leq x+2$

(p2) ... $x-y-3\leq 0$

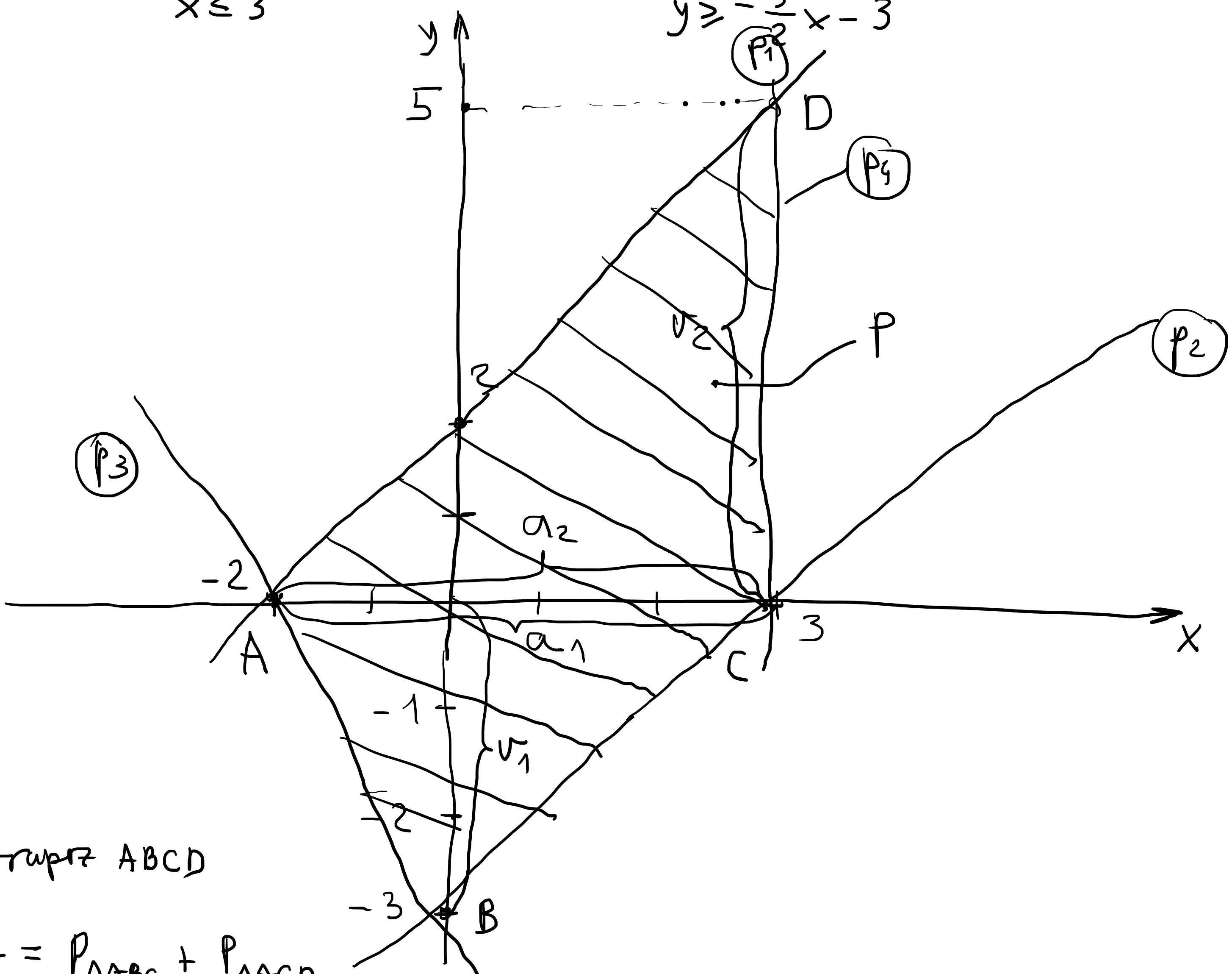
$y\geq x+3$

(p3) ... $x-3\leq 0$

$x\leq 3$

(p4) ... $3x+2y+6\geq 0$

$y\geq -\frac{3}{2}x-3$



trapez ABCD

$$P_T = P_{\Delta ABC} + P_{\Delta ACD}$$

$$P_{ABC} = \frac{a_1 v_1}{2} = \frac{5\cdot 3}{2} = \frac{15}{2}$$

$$P_{\Delta ACD} = \frac{a_2 v_2}{2} = \frac{5\cdot 5}{2} = \frac{25}{2}$$

(+) $\boxed{P_T = \frac{15}{2} + \frac{25}{2} = \frac{40}{2} = 20 \text{ kv. jed.}}$

8 (Pr 11)

$p_1 \cdots 2mx + 3y + 2 = 0 \Rightarrow y = \underbrace{-\frac{2}{3}m}_{k_1} x - \frac{2}{3}$

$p_2 \cdots 2x + my - 1 = 0 \Rightarrow y = \underbrace{-\frac{2}{m}}_{k_2} x + \frac{1}{m}$

a) paralelni pravci: $k_1 = k_2$

$$-\frac{2}{3}m = -\frac{2}{m}$$

$$m^2 = 3$$

$$\boxed{m_{1,2} = \pm\sqrt{3}}$$

b) okomiti pravci: $k_1 = -\frac{1}{k_2}$

$$-\frac{2}{3}m = -\frac{1}{-\frac{2}{m}} = \frac{m}{2} \quad -$$

→ pravci ne mogu biti okomiti?

18. KRUŽNICA

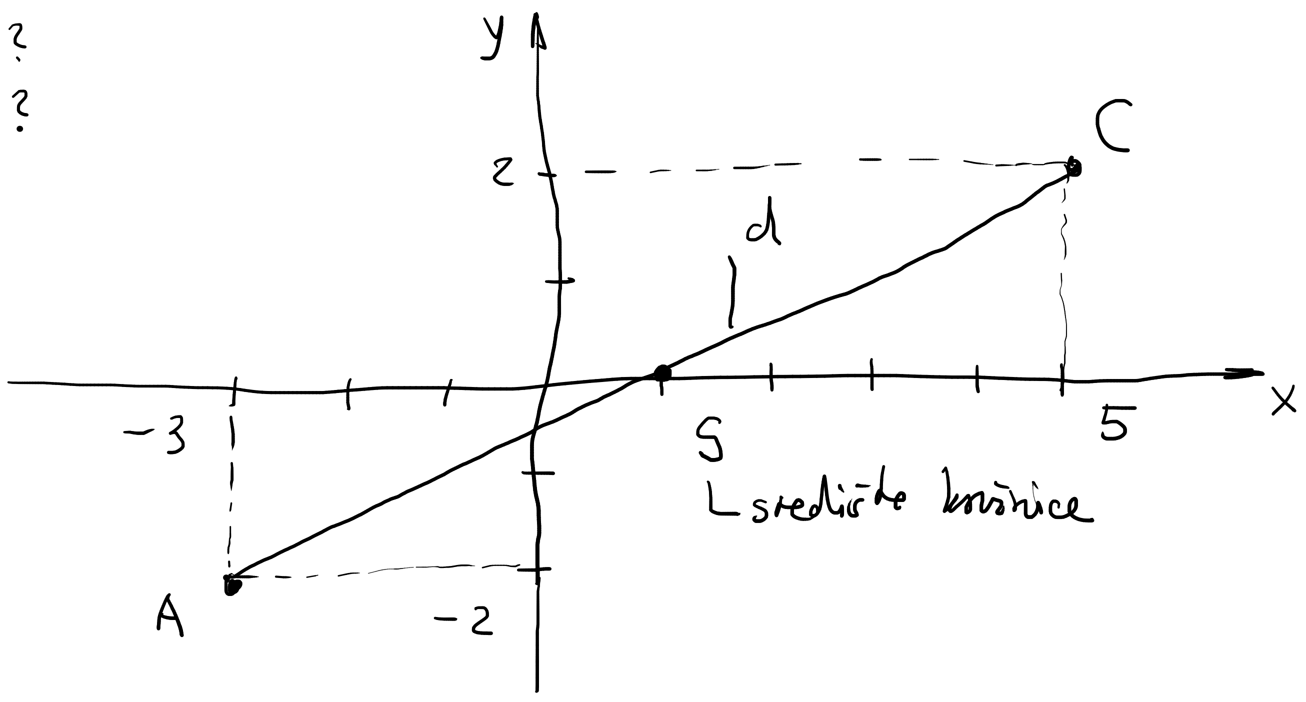
1 (Pr 2)

dijagonala kvadrata $\overline{AC} \cdots A(-3,-2)$

$C(5,2)$

$k_o = ?$

$k_v = ?$



Koord. središta kružnice $S(x_S, y_S)$:

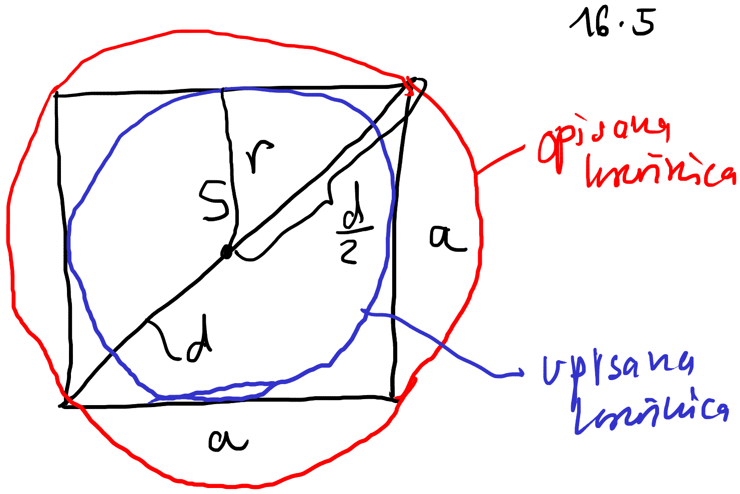
$$x_S = \frac{x_A + x_C}{2} = \frac{-3+5}{2} = \frac{2}{2} = 1$$

$$y_S = \frac{y_A + y_C}{2} = \frac{-2+2}{2} = 0$$

Duljina dijagonale kvadrata d:

$$d = |\overline{AC}| = \sqrt{(y_C - y_A)^2 + (x_C - x_A)^2} = \sqrt{(5-(-3))^2 + (2-(-2))^2}$$

$$= \sqrt{64+16} = \underbrace{\sqrt{80}}_{16\cdot 5} = 4\sqrt{5}$$



$$\frac{d}{2} = R$$

$$r = \frac{a}{2}$$

$$a = ?$$

$$d = \sqrt{a^2 + a^2} = \sqrt{2}\,a$$

$$\Rightarrow a = \frac{\sqrt{2}}{2} d = \frac{\sqrt{2}}{2} \cdot 4\sqrt{5} = 2\sqrt{10}$$

$$r = \frac{2\sqrt{10}}{2} = \sqrt{10}$$

$$R = \frac{d}{2} = \frac{4\sqrt{5}}{2} = 2\sqrt{5}$$

Jednadžba upisane kružnice tom kvadratu:

$$(x - x_S)^2 + (y - y_S)^2 = r^2$$

$$\boxed{(x-1)^2 + y^2 = (\sqrt{10})^2 = 10} \quad \ldots \quad k_U$$

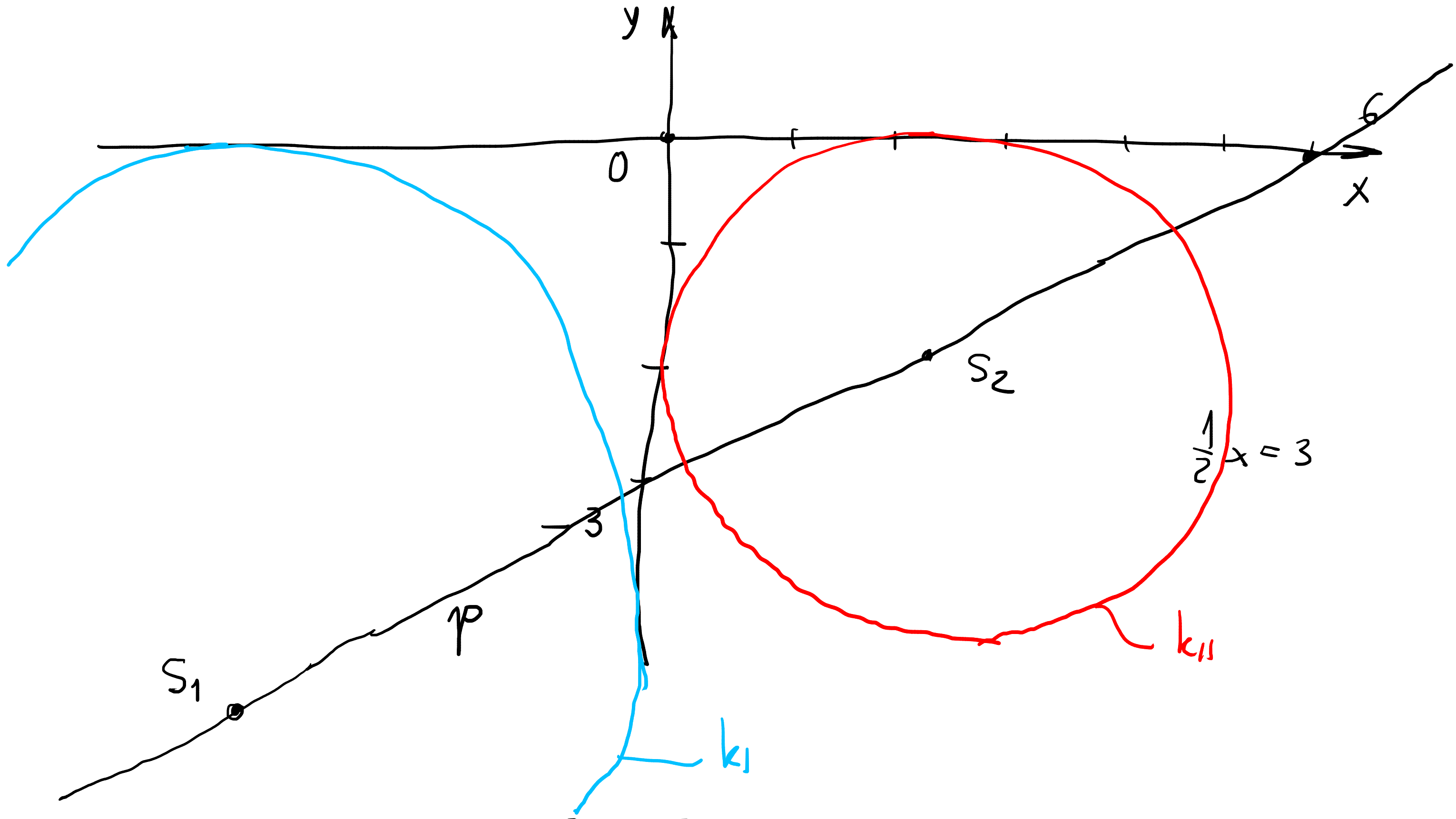
Jednadžba opisane kružnice tom kvadratu:

$$(x - x_S)^2 + (y - y_S)^2 = R^2$$

$$\boxed{(x-1)^2 + y^2 = (2\sqrt{5})^2 = 20} \quad \ldots \quad k_O$$

(2) (Pr 3)

$p \cdots x - 2y - 6 = 0 \Rightarrow y = \frac{1}{2}x - 3$



$k \cdots (x-p)^2 + (y-q)^2 = r^2$

$r = |p| = |q|$ – kad kružnica dira obje koord. osi

$p - 2q - 6 = 0$

$\Rightarrow p = 2q + 6$

$|p| = |q|$

$|p| = 2|q| + 6 = |2q + 6| \rightarrow$ 2 slučaja:

I) $p = 2q + 6$

$q = 2q + 6$

$p = q = -6$

$\boxed{(x+6)^2 + (y+6)^2 = 36} \cdots k_I$

II) $-p = 2q + 6$

$-q = 2q + 6$

$p = q = -2$

$\boxed{(x+2)^2 + (y+2)^2 = 4} \cdots k_{II}$

③ (Pr 5)

(k) ... $x^2+y^2+2x-4y+1=0 \longrightarrow (x+1)^2+(y-2)^2=4$

$S(-1,2)$

k' – koncentrična kružnici k

k' – dira pravac (p) ... $x-3y-3=0$

$S'=S$ – kružnice k' i k su koncentrične

$$d(S,p)=r'=\frac{|Ax_S+By_S+C|}{\sqrt{A^2+B^2}}=\frac{|1\cdot(-1)+(-3)\cdot 2-3|}{\sqrt{1^2+(-3)^2}}=\frac{10}{\sqrt{10}}\cdot\frac{\sqrt{10}}{\sqrt{10}}$$

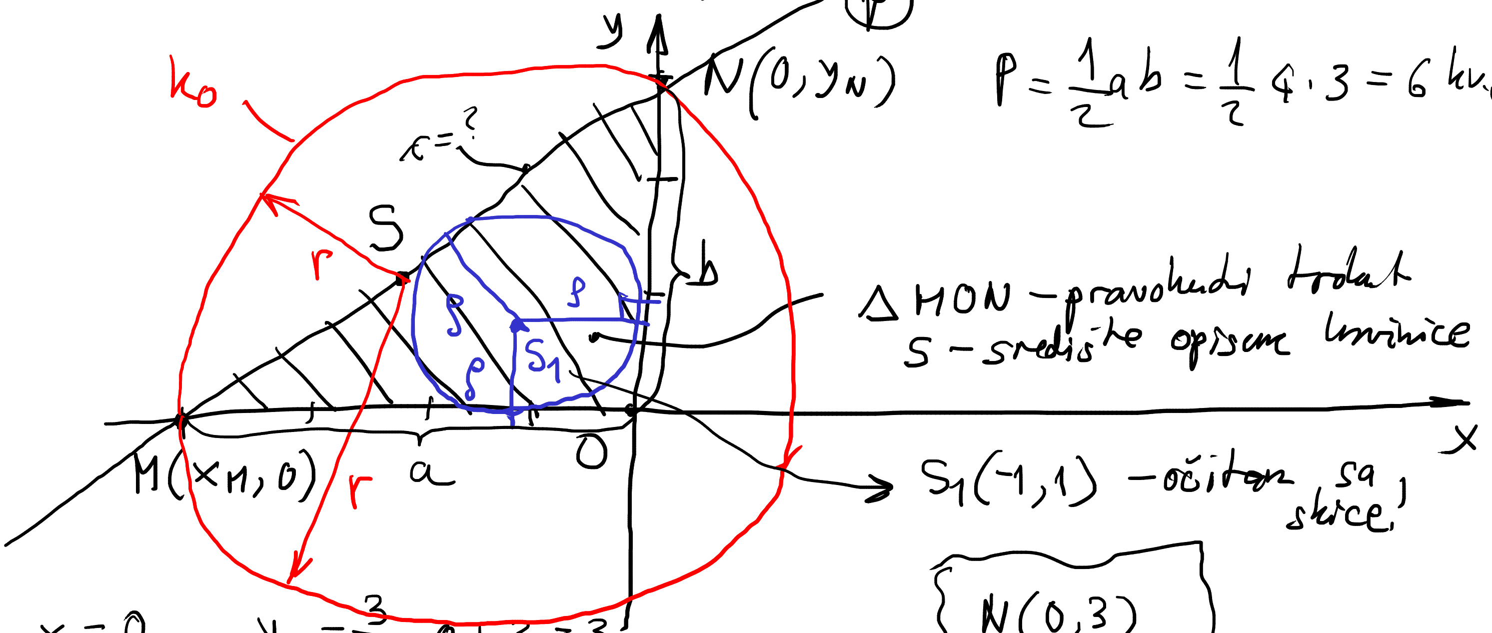
$d=\sqrt{10}$

(k') ... $\boxed{(x+1)^2+(y-2)^2=r'^2=(\sqrt{10})^2=10}$

④ (Pr 6)

(p) ... $3x-4y+12=0 \Rightarrow y=\frac{3}{4}x+3$

– zatvara s osi x i y pravokutni trokut



$P=\frac{1}{2}ab=\frac{1}{2}4\cdot 3=6$ kv. jed.

ΔMON – pravokutni trokut

S – središte opisane kružnice

$S_1(-1,1)$ – očitan sa skice!

$x=0 \qquad y_N=\frac{3}{4}\cdot 0+3=3$

$y=0 \qquad 0=\frac{3}{4}x_M+3 \Rightarrow x_M=-4$

$\boxed{N(0,3) \quad M(-4,0)}$

– korištenjem Tales-ovog poučka:

$$\left.\begin{aligned} x_S&=\frac{x_M+x_N}{2}=\frac{-4+0}{2}=-\frac{4}{2}=-2\\ y_S&=\frac{y_M+y_N}{2}=\frac{0+3}{2}=\frac{3}{2}\end{aligned}\right\} S\left(-2,\frac{3}{2}\right)$$

Radijus opisane kružnice:

$$r = |MS| = \sqrt{(x_S - x_M)^2 + (y_S - y_M)^2} = \sqrt{(-2-(-4))^2 + \left(\frac{3}{2} - 0\right)^2}$$

$$= \sqrt{4 + \frac{9}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

Jednadžba opisane kružnice:

$$(x - x_S)^2 + (y - y_S)^2 = r^2$$

$$k_o \ldots \boxed{(x+2)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{25}{4}}$$

Upisane kružnice:

$\rho = ?$

$$P = \rho \cdot s \Rightarrow \rho = \frac{P}{s}$$

s → poluopseg trokuta

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(4+3+5) = \frac{12}{2} = 6$$

$$\left.\begin{matrix} a = 4 \\ b = 3 \end{matrix}\right\} c = 5$$

$$\rho = \frac{6}{6} = 1$$

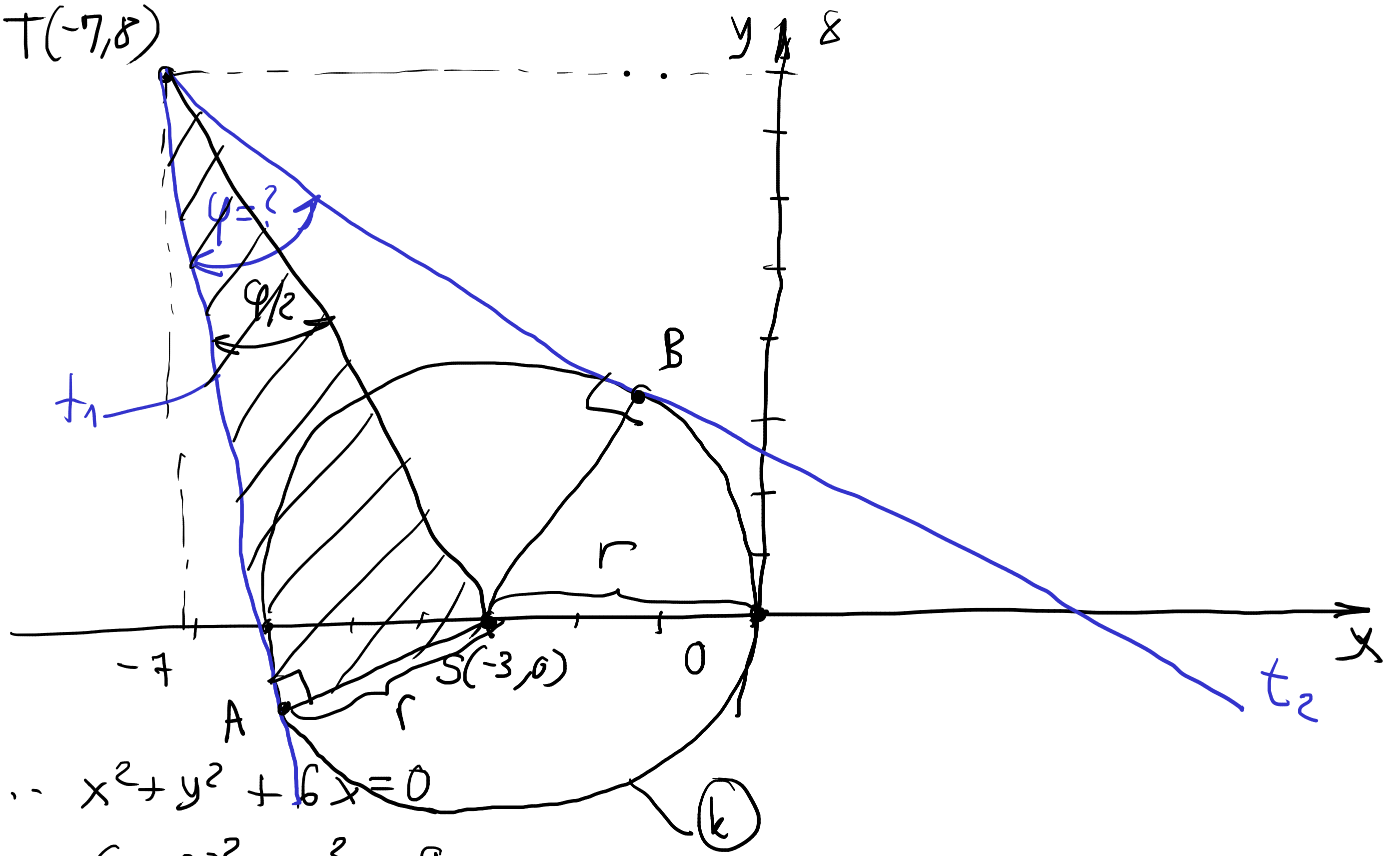
$S_1(x_{S1}, y_{S1})$ – prema skici

$$k_u \ldots \boxed{(x+1)^2 + (y-1)^2 = 1}$$

4 (Pr 7)

$T(-7, 8)$ → pomiče elnje tangente na kružnicu k

k $\ldots x^2 + y^2 + 6x = 0$



$k \, .. \; x^2+y^2+6x=0$

$$\underbrace{(x+3)^2+y^2}_{S(-3,0)} = 9 \to r=3$$

$|\overline{AS}| = r = 3$

$|\overline{ST}| = \sqrt{(y_T-y_S)^2+(x_T-x_S)^2} = \sqrt{(-7-(-3))^2+(8-0)^2} = 4\sqrt{5}$

$$\sin\left(\frac{\varphi}{2}\right) = \frac{r}{|ST|} = \frac{3}{4\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{20} \approx 0{,}335$$

$$\Rightarrow \frac{\varphi}{2} = \arcsin(0{,}335) = 19{,}597^\circ \;/\cdot 2$$

$$\boxed{\varphi = 39{,}195^\circ = 39^\circ 11' 42''}$$

(5)

$P_{kv} = \pi$

(k1) … $x^2+y^2-2x+6y+1=0$

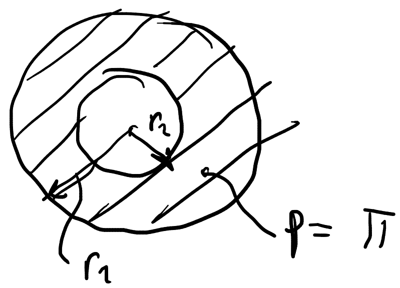
(k2) … ? manja kružnica

$$\underbrace{x^2-2x+1}_{(x-1)^2} + \underbrace{y^2+6y+9}_{(y+3)^2} = -1+1+9 = 9$$

$r=3$

$S(1,-3)$

Kružni vijenac:



$r_2 = ?$

$$P=(r_1^2-r_2^2)\underset{1}{\cancel{\pi}} = \underset{1}{\cancel{\pi}}$$

$$r_1^2-r_2^2=1$$

$$3^2-r_2^2=1$$

$$r_2^2=9-1=8 \quad /\sqrt{\ }$$

$$r_2=2\sqrt{2}$$

(k2) … $\boxed{\begin{array}{l}(x-1)^2+(y+3)^2=8\\ x^2+y^2-2x+6y+2=0\end{array}}$