9. FUNKCIJE

① (Primjer 2)

Ako je $f(1-x)=1-x^2$, za svaki $x\in R$, koliko je $f(1+x)$?

sopstitucija: $t=1-x \Rightarrow x=1-t \ /^2$

$$x^2=1-2t+t^2$$

$$f(t)=1-(1-2t+t^2)=-t^2+2t$$

$$f(1+x)=-(1+x)^2+2(1+x)$$

$$\boxed{f(1+x)=-(1+\cancel{2x}+x^2)+2+\cancel{2x}=-x^2+1}$$

② (Primjer 3)

Ako za svaki $x\in D_f$ vrijedi $\frac{1}{2}f(x)+2f\left(\frac{1}{x}\right)=x$, koliko je $f(2)$?

$$\left.\begin{array}{lll} \frac{1}{2}f(2)+2f\left(\frac{1}{2}\right)=2 & \text{za } x=2 & (1) \\ \frac{1}{2}f\left(\frac{1}{2}\right)+2f(2)=\frac{1}{2} & \text{za } x=2 & (2)\ /\cdot(-4) \end{array}\right\}(+)$$

$$\left.\begin{array}{l} \frac{1}{2}f(2)+\cancel{2f\left(\frac{1}{2}\right)}=2 \\ -\cancel{2f\left(\frac{1}{2}\right)}-8f(2)=-2 \end{array}\right\}(+)$$

$$\left(-8+\frac{1}{2}\right)f(2)=0$$

$$-\frac{15}{2}f(2)=0 \ \Big/ : \left(-\frac{15}{2}\right)$$

$$\boxed{f(2)=0}$$

③ (Primjer 6)
Je li funkcija $f(x) = \dfrac{2^x - 3^x}{2^x + 3^x}$ parna ili neparna ili nije ni parna ni neparna?

$f(x)$ definirana za $\forall x \in \mathbb{R}$

$$f(-x) = \frac{2^{-x} - 3^{-x}}{2^{-x} + 3^{-x}} \Big/ \cdot \frac{2^x \cdot 3^x}{2^x \cdot 3^x}$$

$$= \frac{\underbrace{2^{-x+x}}_{=1} \cdot 3^x - 2^x \cdot \underbrace{3^{x-x}}_{=1}}{\underbrace{2^{-x+x}}_{=1} \cdot 3^x + 2^x \cdot \underbrace{3^{x-x}}_{=1}} = \frac{3^x - 2^x}{3^x + 2^x}$$

$$\underset{\uparrow}{=} -\frac{2^x - 3^x}{2^x + 3^x} = \boxed{-f(x)}$$

izlučim -1

funkcija je neparna!

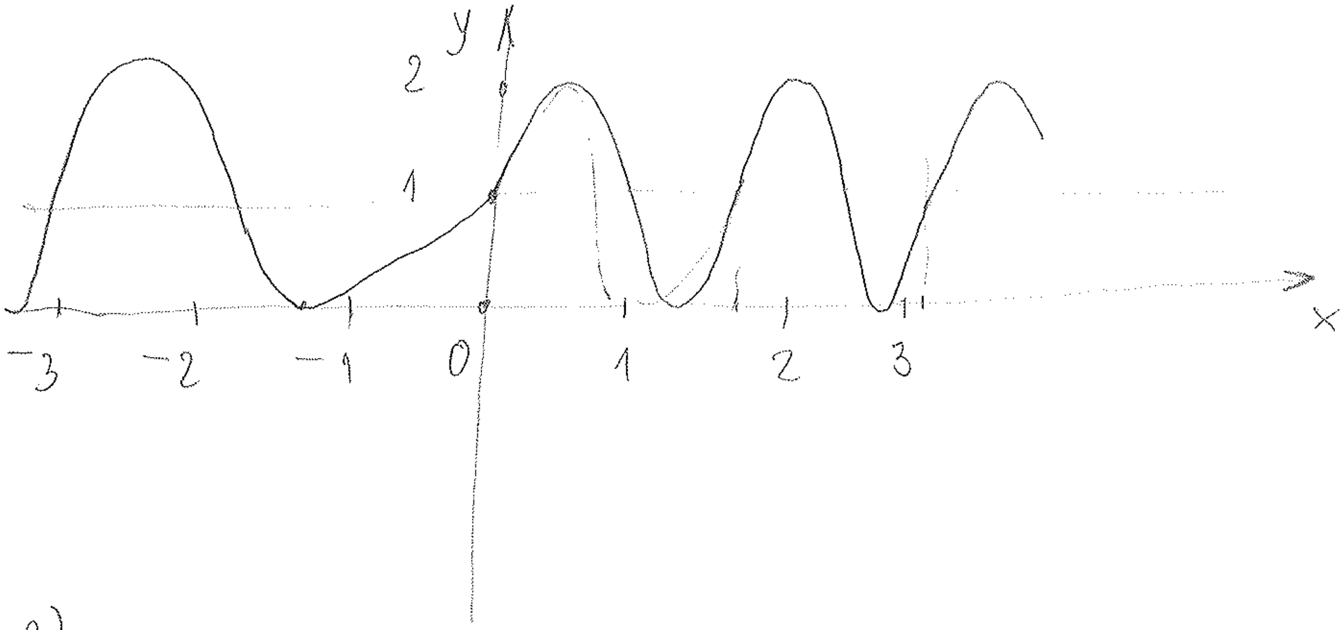
④ (Primjer 8)
Dana je funkcija $f(x) = (\sin 2x + \cos 2x)^2$.
1) Je li ta funkcija parna ili neparna?
2) Je li ta funkcija periodična?
3) Odredimo nultočke ove funkcije.
4) Odredimo skup vrijednosti ove funkcije.

$$f(x) = (\sin 2x + \cos 2x)^2 = \underbrace{\sin^2 2x + \overbrace{2 \cdot \sin 2x \cdot \cos 2x}^{\sin 4x} + \cos^2 2x}_{=1}$$

$$= 1 + \sin 4x$$

$$f(-x) = 1 + \sin 4(-x) = \boxed{1 - \sin 4x}$$

nije ni parna ni neparna!



2)

temeljni period P

$f(x) = a \cdot \sin(bx + c)$

$\to P = \frac{2\pi}{b} = \frac{2\pi}{4} = \frac{\pi}{2}$

amplituda $a = 1$

$c = 0$

periodična funkcija - s $\boxed{P = \frac{\pi}{2}}$

3) nul točke funkcije:

$f(x) = 0$

$1 + \sin 4x = 0$

$\sin 4x = -1 \implies 4x = -\frac{\pi}{2} + k \cdot 2\pi \quad / : 4$

$\boxed{x = -\frac{\pi}{8} + \frac{k\pi}{2}}, \; k \in Z$

4) vrijednosti funkcije: $[0, 2]$

(5) Odredimo inverznu funkciju od
(Pr 11) $$f(x) = \frac{x}{x-2}, \quad x \neq 2$$

$$y = \frac{x}{x-2}$$

$$\Rightarrow \quad x = yx - 2y$$

$$x(1-y) = -2y$$

$$x = \frac{2y}{y-1}$$

$\Rightarrow$ za $f(x)$ mora postojati jedinstveno rješenje
tada f ima svoj inverz $x = f^{-1}(y)$

$$f^{-1}(y) = \frac{2y}{y-1}$$

- zamjena imena nepoznanicama x i y

$$y = f^{-1}(x)$$

$$f^{-1}(x) = \frac{2x}{x-1}$$

(6) Ako je $f(x) = x^2 - x - 1$, a $g(x) = 2x+1$, riješimo
(Pr 9) nejednadžbu $f \circ g(x) \leq g \circ f(x)$.

$$f \circ g(x) = f(g(x)) = (2x+1)^2 - (2x+1) - 1 = 4x^2 + 4x + \cancel{1} - 2x - \cancel{1} - 1 = 4x^2 + 2x - 1$$

$$g \circ f(x) = g(f(x)) = 2(x^2 - x - 1) + 1 = 2x^2 - 2x - 2 + 1 =$$
$$= 2x^2 - 2x - 1$$

$$4x^2 + 2x - \cancel{1} \leq 2x^2 - 2x - \cancel{1}$$

$$2x^2 + 4x \leq 0 \quad / : 2$$

$$x^2 + 2x \leq 0$$

$$x(x+2) \leq 0$$

$$x_1 \leq 0$$
$$x_2 + 2 \leq 0$$

$x_2 \geq -2$

$\boxed{[-2, 0]}$

(7) Za funkciju f vrijedi $f(8^{x-1}) = 4^{x+1}$. Odredimo
(Pr 13) inverznu funkciju od f.

$$4^{x+1} = 4^{x+1-1+1} = 4^{x-1} \cdot 4^2 = 16 \cdot 2^{2(x-1)\frac{3}{3}}$$

$$= 16 \cdot 2^{3(x-1)\frac{2}{3}} = 16 (8^{x-1})^{\frac{2}{3}} = f(8^{x-1})$$

$$f(x) = 16\, x^{\frac{2}{3}}$$

$$x = 16\, y^{\frac{2}{3}} = 16 \cdot \sqrt[3]{y^2}$$

$$x^3 = 16^3 \cdot y^2 \Rightarrow y^2 = \frac{x^3}{16^3}$$

$$y = \left(\frac{x^3}{16^3}\right)^{\frac{1}{2}} = \frac{x^{\frac{3}{2}}}{16^{\frac{3}{2}}}$$

$$y = \frac{x\sqrt{x}}{(2^{\cancel{4}\,2})^{\frac{3}{\cancel{2}\,1}}} = \frac{1}{64} x\sqrt{x}$$

f definirana za sve realne brojeve, $\forall x \in$

$$\boxed{f^{-1}(x) = \frac{1}{64} x\sqrt{x}}$$

8 (Pr 12)

Odredimo inverznu funkciju $f(x) = a \cdot \log_2 x + b$, ako je $f(2) = 0$ i $f(1) = 1$.

$$f(2) = a \cdot \underbrace{\log_2 2}_{=1} + b = 0$$

$$a + b = 0 \quad (1) \quad \Rightarrow \quad a = -1$$

$$f(1) = a \cdot \underbrace{\log_2 1}_{=0} + b = 1 \quad \Rightarrow \quad b = 1 \quad (2)$$

$$f(x) = -\log_2 x + 1$$

$$y = -\log_2 x + 1$$

$$-y + 1 = \log_2 x$$

$$2^{1-y} = x$$

$$\boxed{f^{-1}(x) = 2^{1-x}}$$

9 (zadatak 24 - 2.5 - crvena)

10 (zadatak 59 - 2.1. - crvena)

DZ

zadaci - 1, 3, 16, 17, 18, 19

ispit 1 - 1, 2, 8

ispit 2 - 2, 5, 7

ispit 3 - 2, 4, 6, 8

ispit 4 - 1, 2, 7, 8

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1 (Pr 3)

$$\sin x = \frac{2a+1}{a-2}$$

$$-1 \le \sin x \le 1$$

$$-1 \le \frac{2a+1}{a-2} \le 1$$

1)

$$\frac{2a+1}{a-2} \ge -1$$

$$\frac{2a+1}{a-2} + 1 \ge 0$$

$$\frac{2a+1+1(a-2)}{a-2} \ge 0$$

$$\frac{2a+a+1-2}{a-2} \ge 0$$

$$\frac{3a-1}{a-2} \ge 0$$

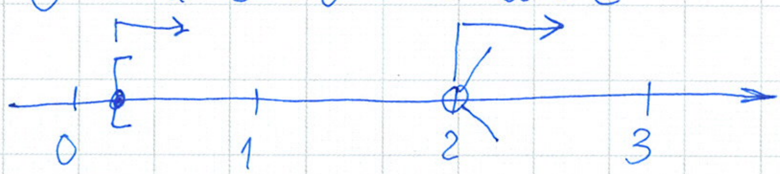
(+) $3a-1 \ge 0 \Rightarrow a \ge \frac{1}{3}$

(+) $a-2 > 0 \Rightarrow a > 2$

ili

(−) $3a-1 \le 0 \Rightarrow a \le \frac{1}{3}$

(−) $a-2 < 0 \Rightarrow a < 2$



$\frac{1}{3} \ge a$ $\quad a > 2$

$$\boxed{a \in \left\langle -\infty, \frac{1}{3}\right] \cup \left\langle 2, +\infty \right\rangle}$$

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II) $\frac{2a+1}{a-2} \leq 1$

$\frac{2a+1}{a-2} - 1 \leq 0$

$\frac{2a+1 - 1(a-2)}{a-2} \leq 0$

$\frac{2a+1 - a+2}{a-2} \leq 0$

$\frac{a+3}{a-2} \leq 0$

$\ominus$ $\oplus$

$a+3 \leq 0 \Rightarrow a \leq -3$

$a-2 > 0 \Rightarrow a > 2$

$\oplus$ $\ominus$

$a+3 \geq 0 \Rightarrow a \geq -3$

$a-2 < 0 \Rightarrow a < 2$

$\boxed{-3 \leq a < 2}$

$\boxed{a_{II} \in [-3, 2\rangle}$

$\boxed{a = a_I \cap a_{II} \in \left[-3, \frac{1}{3}\right]}$

10. TRIGONOMETRIJSKE FUNKCIJE

① (Primjer 3)
Za koje realne brojeve a postoji realan broj x tako da je $\sin x = \frac{2a+1}{a-2}$?

$$-1 \le \frac{2a+1}{a-2} \le 1$$

↳ najveća

↳ najmanje vrijednost koju funkcija $\sin x$ može primiti

⊕ ⊖

⊕ ⊖

Rješavamo nejednadžbe:

$$\frac{2a+1}{a-2} \ge -1 \quad / \cdot \cancel{(a-2)}$$

$\cancel{2a+1 > -a+2}$

$3a > 1$

$$\frac{2a+1}{a-2} + 1 \ge 0$$

$$\frac{2a+1+a-2}{a-2} \ge 0$$

$$\frac{3a-1}{a-2} \ge 0$$

$3a - 1 \ge 0$

$3a \ge 1$

$a \ge \frac{1}{3}$ (?)

$a - 2 > 0$

$a > 2$

$3a-1 \le 0$

$a \le \frac{1}{3}$

$a-2 < 0$

$a < 2$

$$\boxed{a \le \frac{1}{3} \text{ ili } a > 2}$$

$$\frac{2a+1}{a-2} \le 1 \qquad \boxed{-3 \le a < 2}$$

- rješenje - presjek ovih dvaju skupova rješenja:

$$\left[-3, \frac{1}{3}\right]$$

$$\frac{2a+1}{a-2} - 1 \le 0$$

$$\frac{2a+1-(a-2)}{a-2} \le 0 \quad ✓$$

$a - 2 < 0$

$a < 2$

brojnik ≥ 0

nazivnik < 0

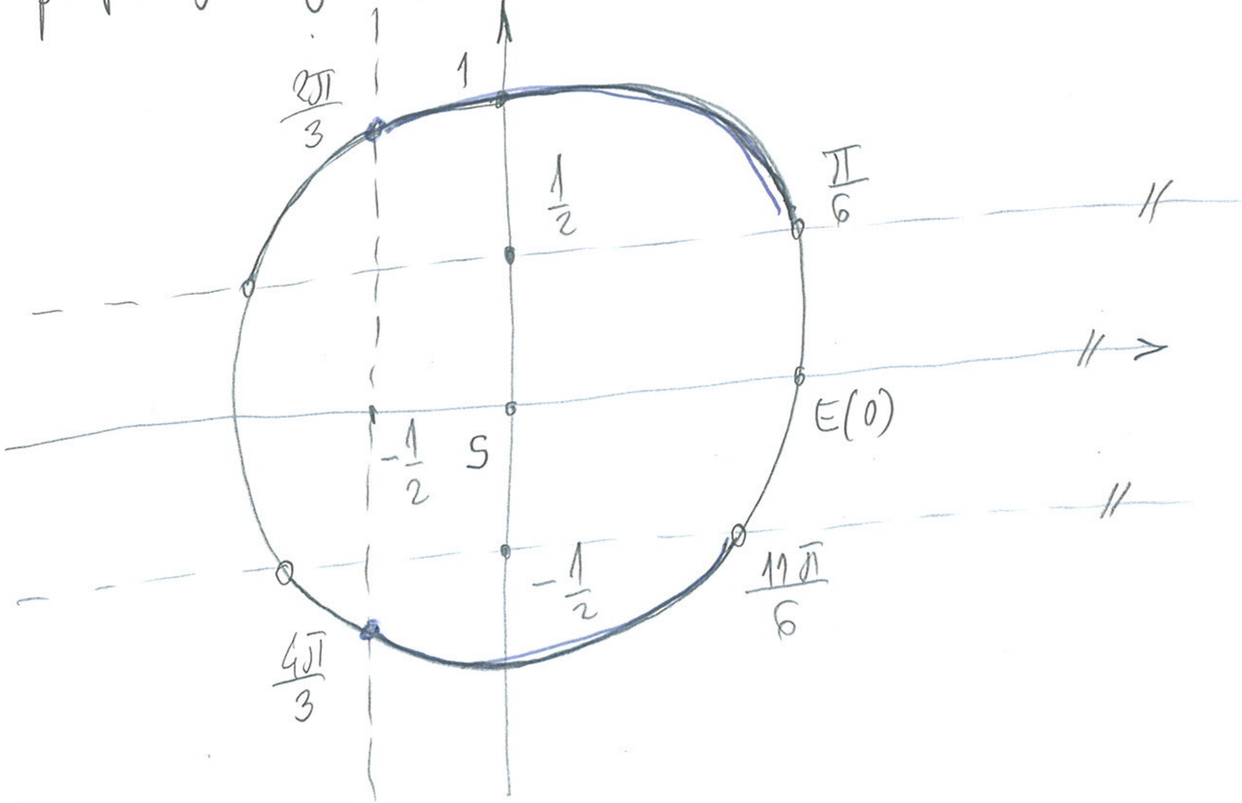
$$\frac{a+3}{a-2} \le 0$$

$a + 3 \ge 0$

$a \ge -3$

$a \ge -3$

② (Primjer 4)
Prikažimo na brojevnoj kružnici skup svih rješenja nejednadžbi $|\sin t| > \frac{1}{2}$ i $\cos t \geq -\frac{1}{2}$. Zapišimo intervale kojima pripadaju rješenja ovog sustava u intervalu $[0, 2\pi]$



1. uvjet: $|\sin t| > \frac{1}{2} \Rightarrow -\sin t > \frac{1}{2}$ ili $\sin t > \frac{1}{2}$

$\sin t < \frac{1}{2}$

2. uvjet: $\cos t \geq -\frac{1}{2}$

skup rješenja: $x \in \left\langle \frac{\pi}{6}, \frac{2\pi}{3} \right] \cup \left[\frac{4\pi}{3}, \frac{11\pi}{6} \right\rangle$

③ (Primjer 5)

Ako je $\sin t + \cos t = \frac{2}{3}$, izračunajmo:

1) $\sin^3 t + \cos^3 t = ?$

$$\sin^3 t + \cos^3 t = (\sin t + \cos t)(\sin^2 t - \sin t \cos t + \cos^2 t) \quad (*)$$

$$(\sin t + \cos t)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

$$\underbrace{\sin^2 t + 2\sin t \cos t + \cos^2 t}_{=1} = \frac{4}{9}$$

$$2\sin t \cos t = \frac{4}{9} - 1 = -\frac{5}{9} \quad / : 2$$

$$\sin t \cos t = -\frac{5}{18}$$

$$\boxed{\sin^3 t + \cos^3 t} = \frac{2}{3}\left(1 - \left(-\frac{5}{18}\right)\right) = \frac{\cancel{2}^{1}}{3} \cdot \frac{23}{\cancel{18}_{9}} = \boxed{\frac{23}{27}}$$

2) $\sin^4 t + \cos^4 t = ?$

$$\boxed{\sin^4 t + \cos^4 t} = \underbrace{(\sin^2 t + \cos^2 t)}_{=1}^2 - 2\sin^2 t \cos^2 t =$$

$$= 1^2 - 2 \cdot \left(-\frac{5}{18}\right)^2 = 1 - \cancel{2}^{1} \cdot \frac{25}{\cancel{324}_{162}} = \boxed{\frac{137}{162}}$$

④ (Primjer 6)

Ako je $\operatorname{tg} t + \operatorname{ctg} t = m$, $m \neq 0$, koliko je:

1) $\operatorname{tg}^3 t + \operatorname{ctg}^3 t = ?$

$$\boxed{\operatorname{tg}^3 t + \operatorname{ctg}^3 t} = (\operatorname{tg} t + \operatorname{ctg} t)^3 - 3\operatorname{tg}^2 t \cdot \operatorname{ctg} t - 3 \operatorname{tg} t \cdot \operatorname{ctg}^2 t$$

$$= \underbrace{(\operatorname{tg} t + \operatorname{ctg} t)^3}_{=m} - 3\operatorname{tg} t(\operatorname{tg} t \operatorname{ctg} t - \operatorname{ctg}^2 t)$$

$$= m^3 - 3\underbrace{\operatorname{tg} t \operatorname{ctg} t}_{=1} \cdot \underbrace{(\operatorname{tg} t + \operatorname{ctg} t)}_{=m}$$

$$= \boxed{m^3 - 3m}$$

2) $\dfrac{1}{\sin^2 t} + \dfrac{1}{\cos^2 t} = ?$

$$\operatorname{tg} t + \operatorname{ctg} t = m$$

$$\frac{\sin t}{\cos t} + \frac{\cos t}{\sin t} = m$$

$$\frac{\overbrace{(\sin^2 t + \cos^2 t)}^{=1}}{\sin t \cdot \cos t} = m$$

$$\frac{1}{\sin t \cos t} = m$$

$$\frac{1}{\sin^2 t} + \frac{1}{\cos^2 t} = \frac{\overbrace{\sin^2 t + \cos^2 t}^{=1}}{\sin^2 t \cdot \cos^2 t} = \frac{1}{\sin^2 t \cos^2 t}$$

$$\boxed{\frac{1}{\sin^2 t} + \frac{1}{\cos^4 t} = m^2}$$

5 (Primjer 9)

Koliko je $\cos(a-b)$, ako je $\sin a + \sin b = 1$, te $\cos a + \cos b = \sqrt{2}$?

$$\sin a + \sin b = 1 \;/^2$$

$$\sin^2 a + 2\sin a \cdot \sin b + \sin^2 b = 1 \qquad (1)$$

$$\cos a + \cos b = \sqrt{2} \;/^2$$

$$\cos^2 a + 2\cos a \cdot \cos b + \cos^2 b = 2 \qquad (2)$$

(1) + (2)

$$\underbrace{\sin^2 a + \cos^2 a}_{=1} + \underbrace{\sin^2 b + \cos^2 b}_{=1} + 2(\overbrace{\sin a \sin b + \cos a \cos b}^{\cos(a-b)}) = 3$$

$$2 + 2\cos(a-b) = 3$$

$$2\cos(a-b) = 3 - 2 = 1$$

$$\boxed{\cos(a-b) = \frac{1}{2}}$$

6 (Primjer 10)

Ako je $\cos(\alpha+\beta) = \frac{1}{3}$, a $\cos(\alpha-\beta) = \frac{1}{5}$, koliko je $\operatorname{tg}\alpha \cdot \operatorname{tg}\beta$?

$$\left.\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta = \frac{1}{3} \\ \cos(\alpha-\beta) &= \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta = \frac{1}{5} \end{aligned}\right\} (1)$$

$$2\cos\alpha \cdot \cos\beta = \frac{8}{15}$$

$$\cos\alpha \cdot \cos\beta = \frac{8}{30} = \frac{4}{15}$$

$$\boxed{\sin\alpha \cdot \sin\beta = \frac{1}{5} - \frac{4}{15} = \frac{3-4}{15} = -\frac{1}{15}}$$

$$\boxed{tg\alpha \cdot tg\beta = \frac{\sin\alpha}{\cos\alpha} \cdot \frac{\sin\beta}{\cos\beta} = \frac{-\frac{1}{15}}{\frac{4}{15}} = -\frac{1}{15} \cdot \frac{15}{4} = -\frac{1}{4}}$$

⑦ (Primjer 13)

Ako je $f\left(x + \frac{3\pi}{2}\right) = \sin x + \cos x$, onda je:

$$f\left(\frac{\pi}{2} - x\right) \cdot f\left(\frac{\pi}{2} + x\right) = \cos 2x.$$

Dokažimo:

$$f\left(x + \frac{3\pi}{2}\right) = \sin x + \cos x = \cos\left(x + \frac{3\pi}{2}\right) - \sin\left(x + \frac{3\pi}{2}\right)$$

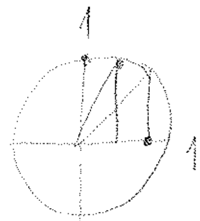
$$\Rightarrow f(x) = \cos x - \sin x$$

$$\boxed{f\left(\frac{\pi}{2} - x\right) \cdot f\left(\frac{\pi}{2} + x\right)} = \left[\cos\left(\frac{\pi}{2} - x\right) - \sin\left(\frac{\pi}{2} - x\right)\right] \cdot \left[\cos\left(\frac{\pi}{2} + x\right) + \sin\left(\frac{\pi}{2} + x\right)\right] =$$

$$= (\sin x - \cos x) \cdot (-\sin x - \cos x) = -\sin^2 x - \cancel{\sin x \cos x} + \cancel{\sin x \cos x} + \cos^2 x = \boxed{\cos(2x)}.$$

⑧ Bez uporabe džepnog kalkulatora izračunajmo:

(Pr 16) 2) $\sin\left(\frac{17\pi}{24}\right) \cdot \sin\left(\frac{23\pi}{24}\right) = ?$



$$\Gamma \quad \sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

$$\sin\left(\frac{17\pi}{24}\right) \cdot \sin\left(\frac{23\pi}{24}\right) = \frac{1}{2}\left[\cos\left(\frac{17\pi - 23\pi}{24}\right) - \cos\left(\frac{17\pi + 23\pi}{24}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{\cancel{6}^{1}\pi}{\cancel{24}_{4}}\right) - \cos\left(\frac{\cancel{40}\pi}{\cancel{24}}\right)\right] = \frac{1}{2}\left[\cos\frac{\pi}{4} - \underbrace{\cos\frac{5\pi}{3}}_{=\frac{\pi}{3}}\right] = \frac{1}{2}\left[\frac{\sqrt{2}}{2} - \frac{1}{2}\right]$$

$$= \boxed{\frac{\sqrt{2} - 1}{4}}$$

DZ

zadaci – 7, 11, 19, 23, 25

ispit 1 – 7, 9

ispit 2 – 10

ispit 3 – 5, 6, 9

ispit 4 – 4, 5, 9

9 (Pr 15)

$$2\,ctg^2x + 7\,ctg\,x + 3 = 0$$

$$\frac{3\pi}{2} < x < \frac{7\pi}{4}$$

$$\cos 2x = ?$$

$ctg\,x = t$ supstitucija

$$2t^2 + 7t + 3 = 0$$

$$t_{1,2} = \frac{-7 \pm \sqrt{49 - 4\cdot 2\cdot 3}}{2\cdot 2}$$

$$t_{1,2} = \frac{-7 \pm 5}{4}$$

$t_1 = -3$

$t_2 = -\frac{1}{2}$

$ctg\,x_1 = -3$ ⊖

$ctg\,x_2 = -\frac{1}{2}$ ✓

→ gledajući prema ovom odgovara samo ovo rješenje.

$$\cos x = \frac{ctg\,x}{\sqrt{1+ctg^2x}} = \frac{-\frac{1}{2}}{\sqrt{1+\left(-\frac{1}{2}\right)^2}} = \frac{-\frac{1}{2}}{\sqrt{1+\frac{1}{4}}} = -\frac{1}{\cancel{2}_1}\cdot\frac{1}{\frac{\sqrt{5}}{\cancel{2}_1}} = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$

$$\sin x = -\sqrt{1-\cos^2 x}$$

$$\sin x = -\sqrt{1-\left(\frac{-\sqrt{5}}{5}\right)^2} = -\sqrt{1-\frac{\cancel{5}}{\cancel{25}_5}} = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= \left(-\frac{\sqrt{5}}{5}\right)^2 - \left(-\frac{2\sqrt{5}}{5}\right)^2 = \frac{5}{25} - \frac{4\cdot 5}{25} = -\frac{15}{25} = \boxed{-\frac{3}{5}}$$

11. TRIGONOMETRIJSKE JEDNADŽBE I NEJEDNADŽBE

① (Primjer 2)

Prikažimo grafički funkciju $f(x) = 2\cos\left(3x - \frac{3\pi}{4}\right) - 1$

spustimo ga za 1 niz os y

$$f(x) = 2\cos\left(3x - \frac{3\pi}{4}\right) - 1$$

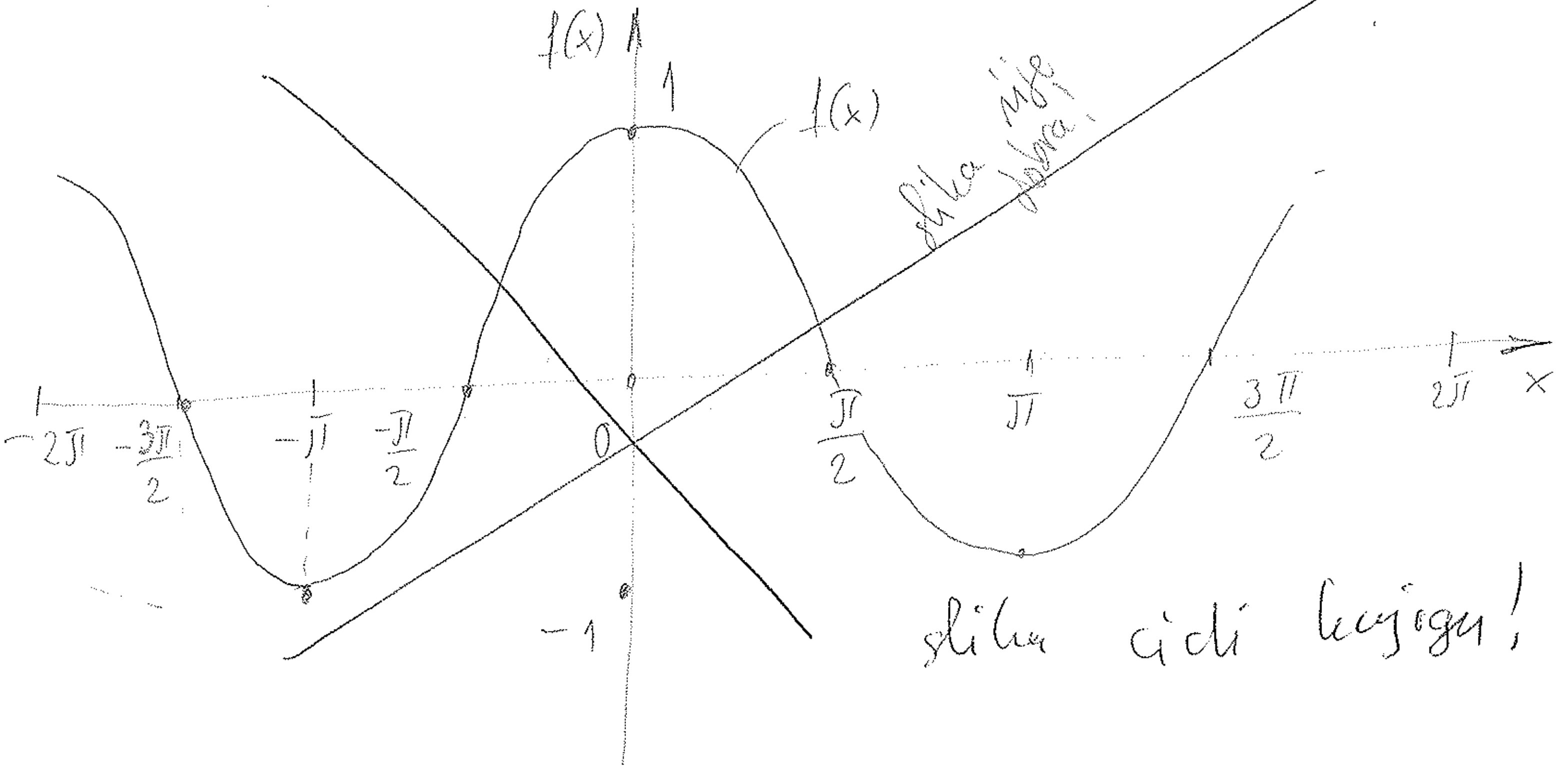
$|a| = 2$ – amplituda funkcije

$$P = \frac{2\pi}{b} = \frac{2\pi}{3}$$

↳ temeljni period

$-\frac{c}{b}$ – pomak grafa funkcije u ⊕ smjeru osi x

$$-\frac{c}{b} = -\frac{-\frac{3\pi}{4}}{3} = \frac{\pi}{4}$$

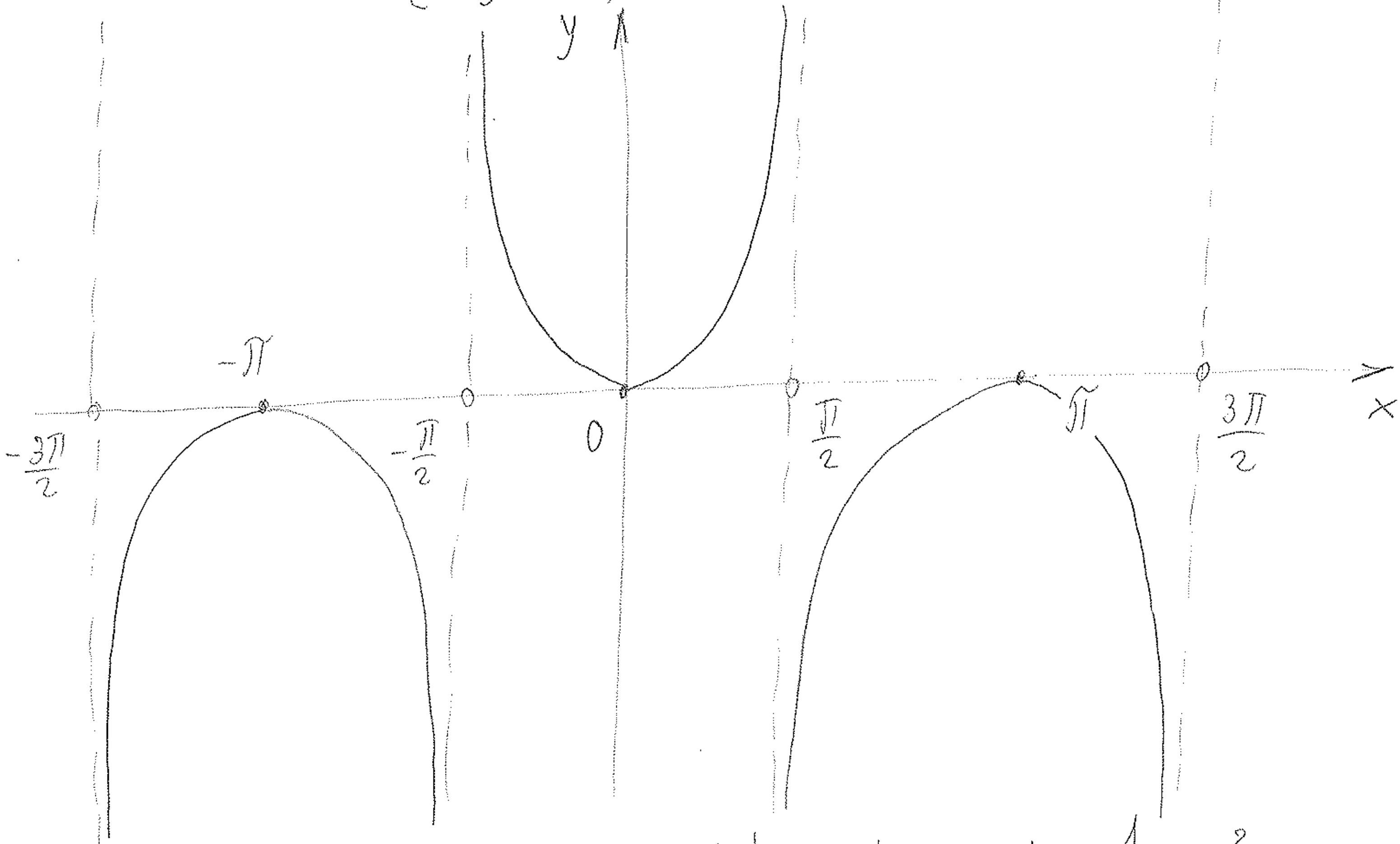


slika cijeli knjigu!

② (Primjer 4)
Prikažimo grafički funkciju $f(x) = \dfrac{\sqrt{1-\cos^2 x}}{\cos x}$.

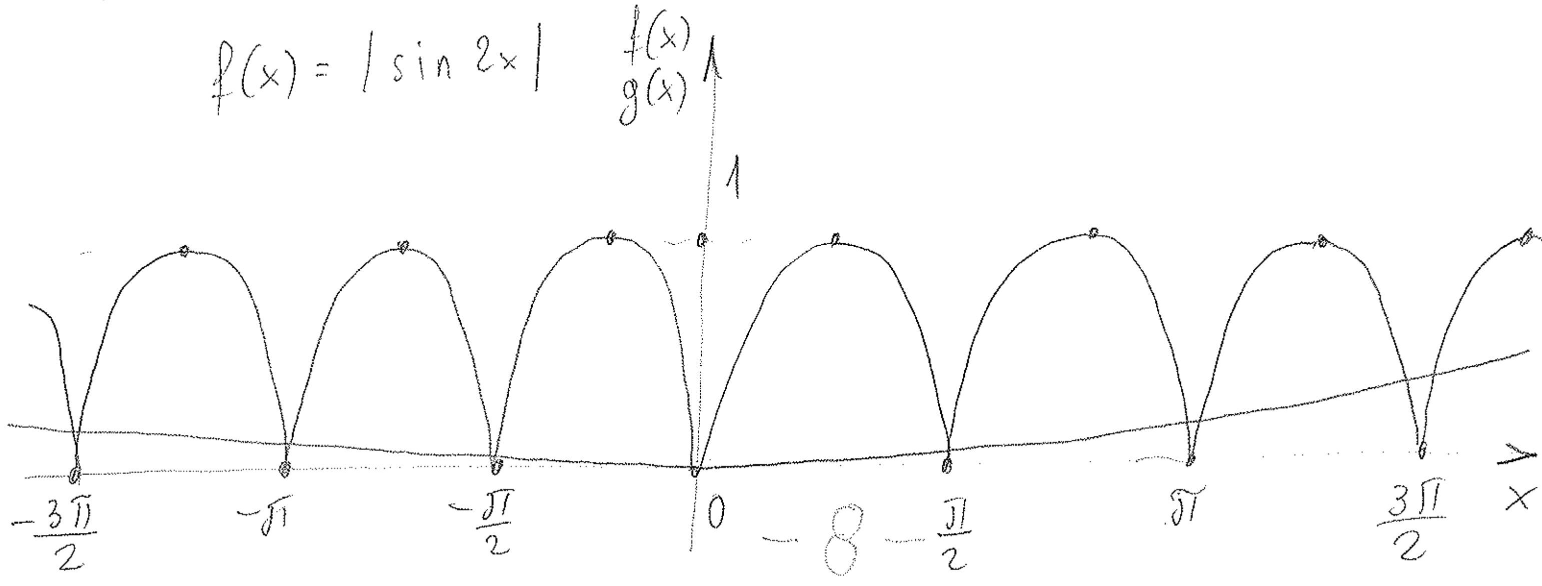
$$f(x) = \frac{\sqrt{1-\cos^2 x}}{\cos x} = \frac{\sqrt{\sin^2 x}}{\cos x} = \frac{|\sin x|}{\cos x}$$

$$f(x) = \begin{cases} \operatorname{tg} x, & \sin x \geq 0 \\ -\operatorname{tg} x, & \sin x < 0 \end{cases}$$



③ Koliko rješenja ima jednadžba $\underbrace{|\sin 2x|}_{f(x)} = \underbrace{\frac{1}{100} x^2}_{g(x)}$.
(Pr 6)

$f(x) = |\sin 2x|$



$\frac{1}{100}x^2 > 1$

$x^2 > 100$

$x > 10$

$\frac{x}{\frac{\pi}{2}} = \frac{10}{\frac{\pi}{2}} = 6,37$

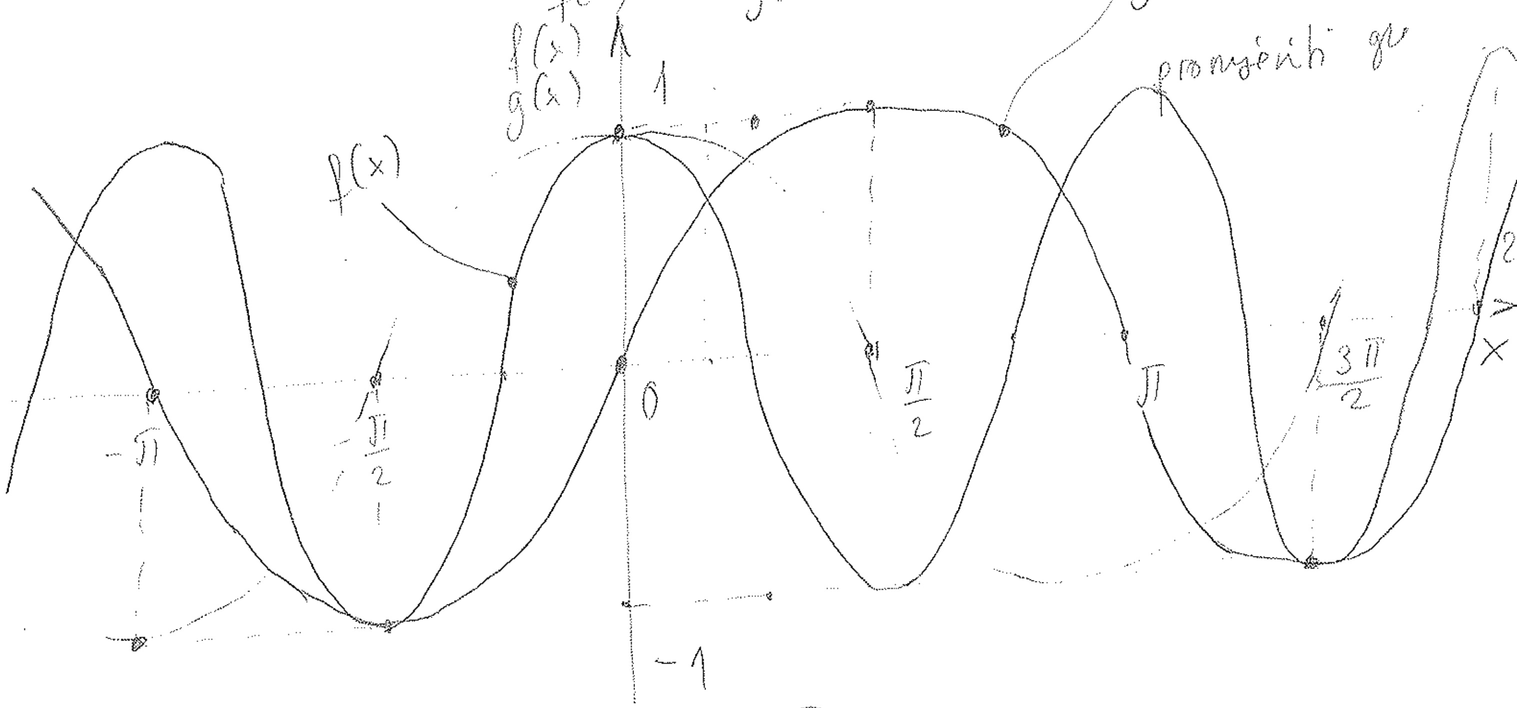
↳ gledamo samo cijeli broj

$n = 6 \cdot 2 \cdot 2 - \underbrace{1}_{} = 23$ puta

↳ jer je u $x=0$ spaja parabola

(4) (Primjer 17)

Riješimo nejednadžbu $\underbrace{\cos 2x}_{f(x)} \le \underbrace{\sin x}_{g(x)}$ na intervalu $[0, 2\pi]$.



za $f(x)$ $P = \frac{2\pi}{2} = \pi$

$2\sin^2 x + \sin x - 1 = 0$

$t = \sin x$

$2t^2 + t - 1 = 0 \qquad t_{1,2} = \frac{-1 \pm \sqrt{1+8}}{4}$ ↗ $t_1 =$ ↘ $t_2 =$

$\sin x = -1 \Rightarrow x = \frac{3\pi}{2}$; $f(x) > g(x)$ (−)

$\sin x = \frac{1}{2} \Rightarrow x_1 = \frac{\pi}{6}$, $x_2 = \frac{5\pi}{6}$ } (+)

5 (Primjer 8)

2) $\sqrt{3} \cdot ctg\left(x - \frac{\pi}{6}\right) = 1 \quad / : \sqrt{3}$

$ctg\left(x - \frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\Rightarrow x - \frac{\pi}{6} = \frac{\pi}{3} + k \cdot \pi$

$x = \frac{\pi}{3} + \frac{\pi}{6} + k \cdot \pi = \frac{2\pi + \pi}{6} + k \cdot \pi$

$\boxed{x = \frac{\pi}{2} + k \cdot \pi}, \quad k \in Z$

6 (Primjer 10)

Riješimo jednadžbu: $2\cos^2 x + 3\sin x \cdot \cos x + 3\sin^2 x = 1$

$\cos^2 x + \sin^2 x = 1$

$2\cos^2 x + 3\sin x \cdot \cos x + 3\sin^2 x = \cos^2 x + \sin^2 x$

$\cos^2 x + 2\sin^2 x + 3\sin x \cdot \cos x = 0 \quad / : \sin^2 x$

$ctg^2 x + 2 + 3\, ctg\, x = 0$

$ctg\, x = t$

$t^2 + 3t + 2 = 0$

$t_{1,2} = \frac{-3 \pm \sqrt{9 - 8}}{2} = \frac{-3 \pm 1}{2}$ → $t_1 = -2$, $t_2 = -1$

za $t_1 = -2 \Rightarrow ctg\, x = -2$

$x_1 \approx -0,46 + k \cdot \pi, \quad k \in Z$

za $t_1 = -1 \Rightarrow ctg\, x = -1$

$x_2 = \frac{3\pi}{4} + k \cdot \pi$

(17) (Primjer 13)

Riješimo jednadžbu: $3\sin x + 2\cos x = 3$.

tzv. trigonometrijske funkcije dvostrukog argumenta
$$\left\{\begin{aligned} \sin x &= \frac{2\,\mathrm{tg}\,\frac{x}{2}}{1+\mathrm{tg}^2\left(\frac{x}{2}\right)} \\ \cos x &= \frac{1-\mathrm{tg}^2\left(\frac{x}{2}\right)}{1+\mathrm{tg}^2\left(\frac{x}{2}\right)} \end{aligned}\right\} \quad \mathrm{tg}\left(\frac{x}{2}\right) = t$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$3\cdot\frac{2t}{1+t^2} + 2\,\frac{1-t^2}{1+t^2} = 3 \quad /\cdot(1+t^2)$$

$$6t + 2 - 2t^2 = 3 + 3t^2$$

$$-5t^2 + 6t - 1 = 0 \quad /\cdot(-1)$$

$$5t^2 - 6t + 1 = 0$$

$$t_{1,2} = \frac{6 \pm \sqrt{36-20}}{10} \quad \begin{aligned} &\nearrow t_1 = 1 \\ &\searrow t_2 = \frac{1}{5} \end{aligned}$$

$$t_1 = \mathrm{tg}\left(\frac{x_1}{2}\right) = 1 \Rightarrow \frac{x_1}{2} = \frac{\pi}{4} + k\cdot\pi \quad /\cdot 2$$

$$\boxed{x_1 = \frac{\pi}{2} + k\cdot 2\pi}$$

$$t_2 = \mathrm{tg}\left(\frac{x_2}{2}\right) = \frac{1}{5} = 0{,}2 \Rightarrow \frac{x_2}{2} = \mathrm{arctg}(0{,}2) + k\cdot 2\pi \quad /\cdot 2$$

$$\boxed{x_2 = 2\,\mathrm{arctg}(0{,}2) + k\cdot 2\pi}$$

8 (Primjer 18)

Riješimo na intervalu $\langle 0, 2\pi \rangle$ nejednadžbu: $\sin x + \sqrt{3}\cos x > 0$.

$$\sin x + \sqrt{3}\cos x > 0$$

$$\sin x > -\sqrt{3}\cos x \quad (1)$$

I) $\cos x > 0$

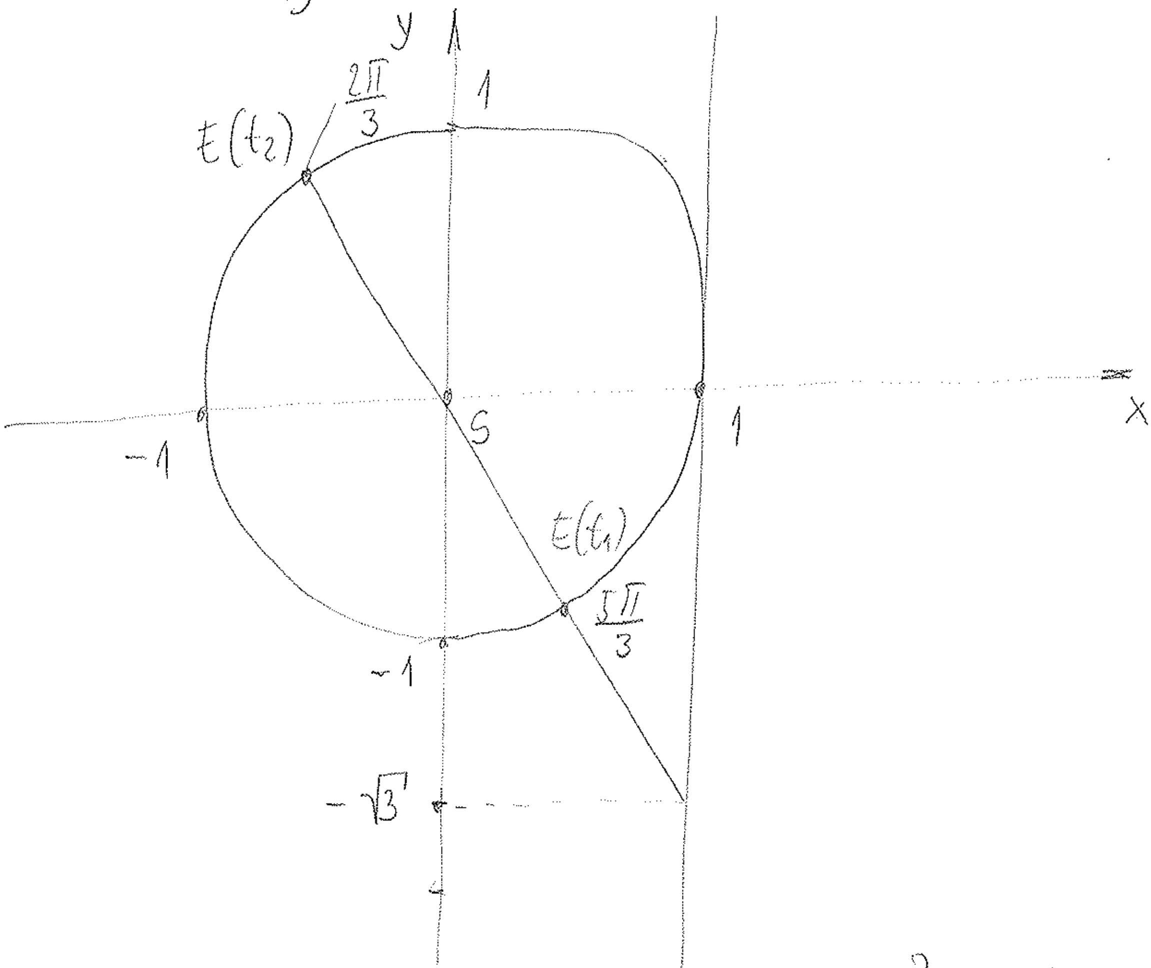
$(1) \; / : \cos x$

$\operatorname{tg} x > -\sqrt{3}$

II) $\cos x < 0$

$(1) \; / : \cos x$

$\operatorname{tg} x < -\sqrt{3}$



I) $\boxed{x_1 \in \left\langle 0, \frac{\pi}{2} \right\rangle \cup \left\langle \frac{5\pi}{3}, 2\pi \right\rangle}$

II) $\boxed{x_2 \in \left\langle \frac{\pi}{2}, \frac{2\pi}{3} \right\rangle}$

unija ovih dvaju rješenja!

$\boxed{x \in \left\langle 0, \frac{2\pi}{3} \right\rangle \cup \left\langle \frac{5\pi}{3}, 2\pi \right\rangle}$

⑨ (Primjer 19)

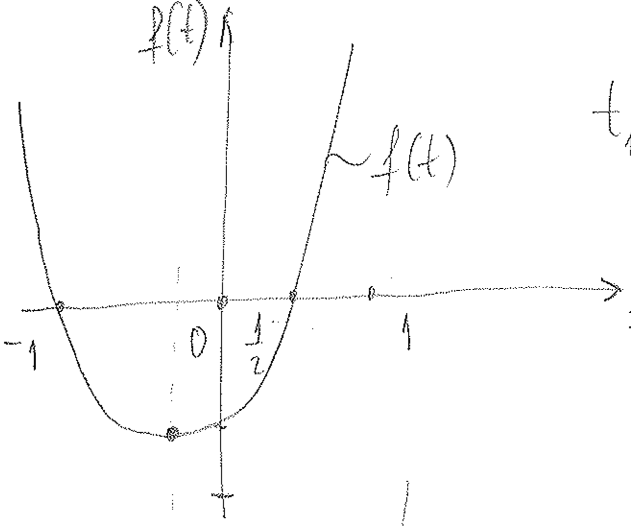
Riješimo nejednadžbu: $2\cos^2 x + \cos x \geq 1$

$$2\cos^2 x + \cos x - 1 \geq 0$$

$$t = \cos x$$

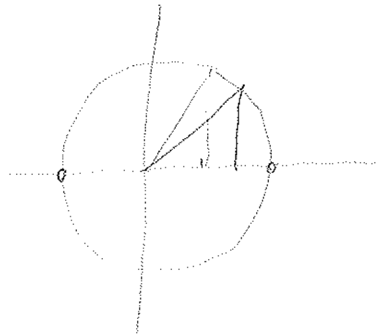
$$2t^2 + t - 1 \geq 0$$

$$t_{1,2} = \frac{-1 \pm \sqrt{1 - 4 \cdot 2 \cdot (-1)}}{4} = \frac{-1 \pm 3}{4} \begin{matrix} \nearrow t_1 = - \\ \searrow t_2 = \frac{1}{2} \end{matrix}$$



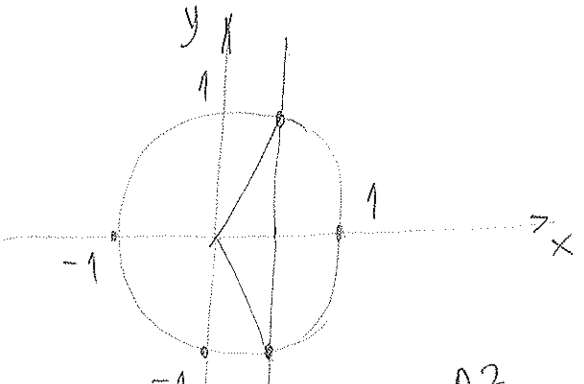
za koje t $\quad f(t) \geq 0$

$$t = \cos x$$



ili $\quad \cos x \geq \frac{1}{2}$

$\quad\quad \cos x = -1$



$$\Rightarrow \boxed{-\frac{\pi}{3} + k \cdot 2\pi \leq x \leq \frac{\pi}{3} + k \cdot 2\pi}$$

<u>DZ</u>

zadaci - 7, 8, 9, 14, 15, 19, 26

ispit 1 - 8, 9

ispit 2 - 2, 10

ispit 3 - 9, 10

12. PLANIMETRIJA

① (Primjer 2)
Za kutove trokuta α, β, γ, vrijedi $\alpha : \beta = 1 : 2$ i $\beta : \gamma = 4 : 9$. Koliki su ti kutovi?

$$\alpha + \beta + \gamma = 180^\circ$$

$$\frac{\alpha}{\beta} = \frac{1}{2} \Rightarrow \beta = 2\alpha$$

$$\frac{\beta}{\gamma} = \frac{4}{9} \Rightarrow \gamma = \frac{9}{4}\beta = \frac{9}{4} \cdot 2\alpha = \frac{9}{2}\alpha$$

$$\alpha + 2\alpha + \frac{9}{2}\alpha = 180^\circ$$

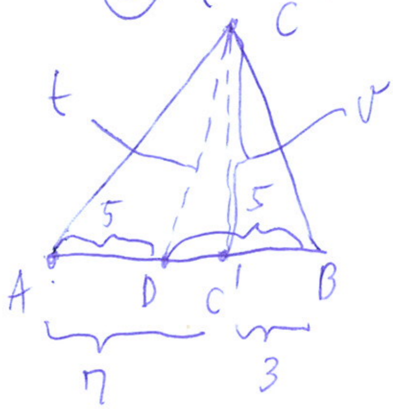
$$\frac{15}{2}\alpha = 180^\circ \quad / \cdot \frac{2}{15}$$

$$\boxed{\alpha = 24^\circ}$$

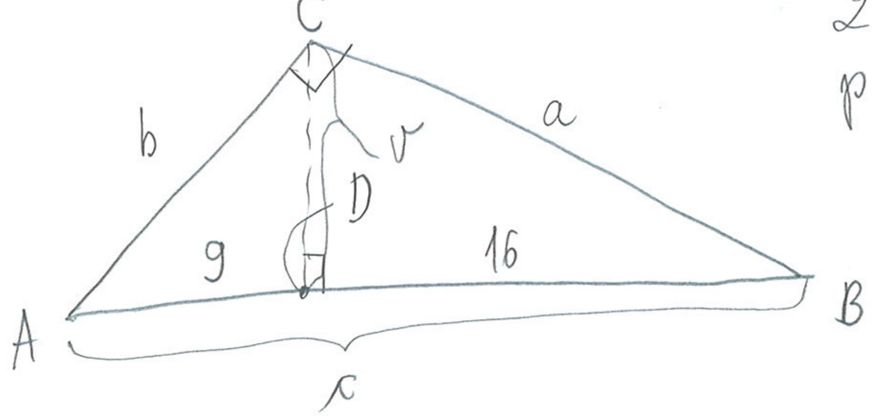
$$\boxed{\beta = 2 \cdot 24 = 48^\circ}$$

$$\boxed{\gamma = \frac{9}{2} \cdot 24 = 108^\circ}$$

② (Pr 3)



② (Primjer 5)
Visina pravokutnog trokuta spuštena je iz vrha pravog kuta te dijeli hipotenuzu na dva dijela s duljinama 16 i 9. Koliki je zbroj duljina kateta toga trokuta?



$q = \overline{AD} = 9$ cm
$p = \overline{BD} = 16$ cm

$c = 9 + 16 = 25$ cm

Euklidov poučak:

$a = \sqrt{c \cdot p} = \sqrt{16 \cdot 25}$

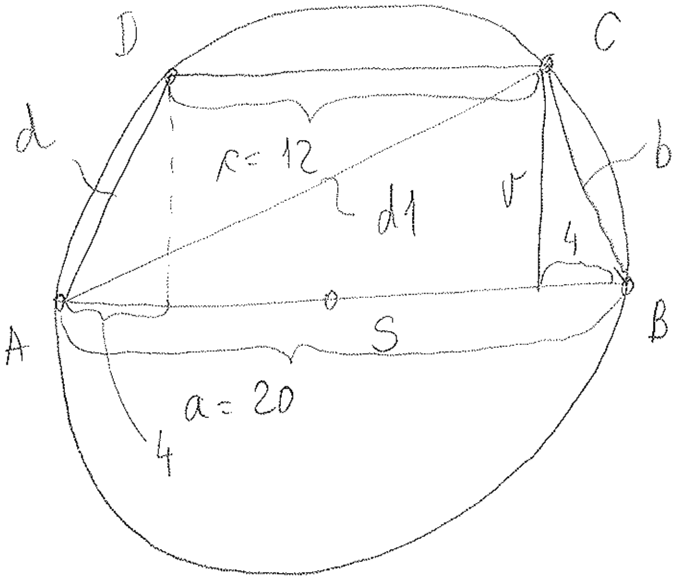
$a = 20$ cm

$b = \sqrt{c \cdot q} = \sqrt{9 \cdot 25}$
$= 15$ cm

$$\boxed{a + b = 20 + 15 = 35 \text{ cm}}$$

③ (Primjer 8)

Duljine osnovice trapeza jednake su 12 cm i 20 cm. Središte kružnice opisane trapezu nalazi se na većoj osnovici. Kolika je duljina kraka i kolika je površina trapeza? Kolika je duljina kraka i kolike su duljine dijagonala trapeza?



$a = \overline{AB} = 20$ cm

$c = \overline{CD} = 12$ cm

1) $P = ?$

2) $b, d = ?$

$d1 = ?$

1) Površina trapeza:

$$P = \frac{a+c}{2} \cdot v$$

$v = \sqrt{16 \cdot 4} = 8$ cm (po Euklidovom poučku)

$$\boxed{P = \frac{20+12}{2} \cdot 8 = 16 \cdot 8 = 128 \text{ cm}^2}$$

2) Duljine krakova:

$$\boxed{b = d = \sqrt{v^2 + 4^2} = \sqrt{8^2 + 4^2} = \sqrt{64+16} = \sqrt{80} = 4\sqrt{5} \text{ cm}}$$

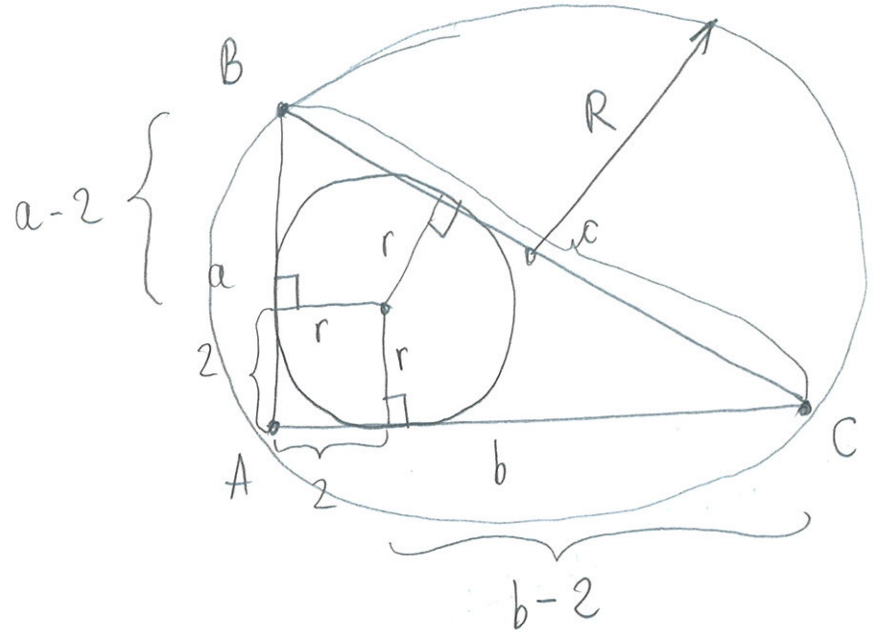
$$\boxed{d1 = \sqrt{8^2 + 16^2} = \sqrt{64+256} = \sqrt{320} = 8\sqrt{5} \text{ cm}}$$

④ (Primjer 9)

Polumjer pravokutnom trokutu upisane kružnice jednak je 2 cm, polumjer opisane kružnice 5 cm. Kolike su duljine stranica ovoga trokuta?

$r = 2$ cm
$R = 5$ cm

$a, b, c = ?$



$c = 2R = (b-2) + (a-2) = 2 \cdot 5$

$\underline{a+b} = 5 \cdot 2 + 4 = \underline{14}$ (1)

Pitagorin poučak: $c^2 = \underline{a^2 + b^2 = 100}$ (2)

iz (1) : $a = 14 - b \rightarrow (2)$

$(14-b)^2 + b^2 = 100$

$196 - 28b + b^2 + b^2 = 100$

$2b^2 - 28b + 96 = 0 \quad / :2$

$\underline{b^2 - 14b + 48 = 0}$

$b_{1,2} = \dfrac{14 \pm \sqrt{(-14)^2 - 4 \cdot 1 \cdot 48}}{2}$ → $b_1 = 6$ cm, $b_2 = 8$ cm

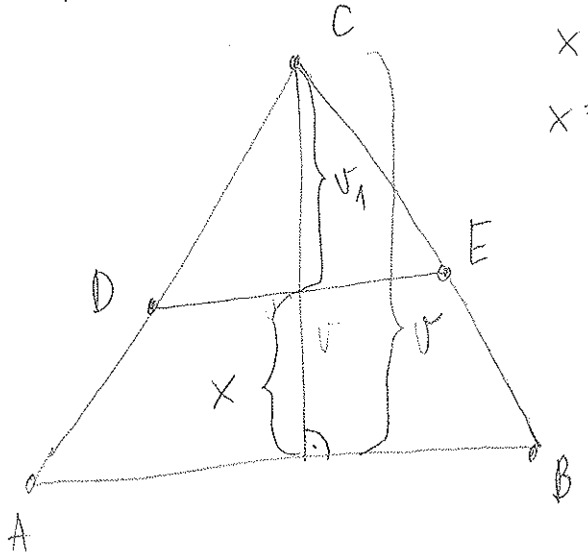
pretpostavka: $b > a \Rightarrow b = b_2 = 8$ cm $\rightarrow (1)$

$\boxed{a} = 14 - 8 = \boxed{6 \text{ cm}}$

⑤ (Pr 11)
Sličnost trokuta

⑤ (Primjer 12)

Visina trokuta spuštena na stranicu $\overline{AB}$ trokuta ΔABC duga je 10 cm. Na kojoj udaljenosti od stranice $\overline{AB}$ treba položiti paralelu s tom stranicom kako bi trokut njome bio podijeljen na dva dijela jednakih površina?



$x = ?$

$x = v - v_1$

$P \cdots \Delta ABC$

$v = 10$ cm

primjetimo: sličnost trokuta $\Delta ABC \sim \Delta CDF$

$P_1 \cdots \Delta CDF$

$P_1 = \frac{P}{2}$

sličnost trokuta: - koeficijent sličnosti trokuta k

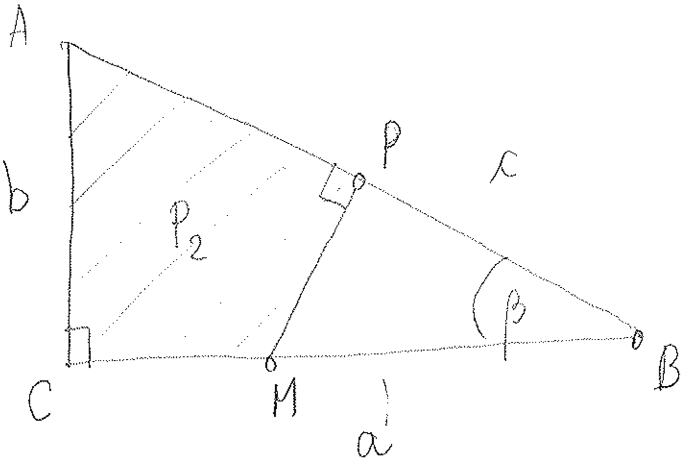
$$k^2 = \frac{P'}{P} = \frac{P}{P_1} = 2 \Rightarrow k = \sqrt{2}$$

$$k = \frac{v}{v_1} = \frac{10}{v_1} \Rightarrow v_1 = \frac{10}{k} = \frac{10}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$v_1 = 5\sqrt{2} \text{ cm}$$

$$\boxed{x} = 10 - 5\sqrt{2} = \boxed{5(2-\sqrt{2}) \text{ cm}}$$

⑥ (Primjer 14)
Katete pravokutnog trokuta iznose $a=5$ i $b=3$. Okomica povu-
čena polovištem hipotenuze rastavlja taj trokut na jedan
četverokut i jedan trokut. Kolika je površina tako dobivenog
četverokuta?



sličnost trokuta:

$\Delta ABC \sim \Delta BMP$

$P \qquad P_1$

$P_2 \ldots \square APMC$

$P_2 = P - P_1$

$$P = \frac{a \cdot b}{2} = \frac{5 \cdot 3}{2} = \frac{15}{2}\ cm$$

pitagorin poučak: $c = \sqrt{a^2+b^2} = \sqrt{5^2+3^2} = \sqrt{34}\ cm$

$$\overline{BP} = \frac{c}{2} = \frac{\sqrt{34}}{2}\ cm$$

prema sličnosti trokuta: $\dfrac{\overline{PB}}{\overline{MP}} = \dfrac{a}{b}$

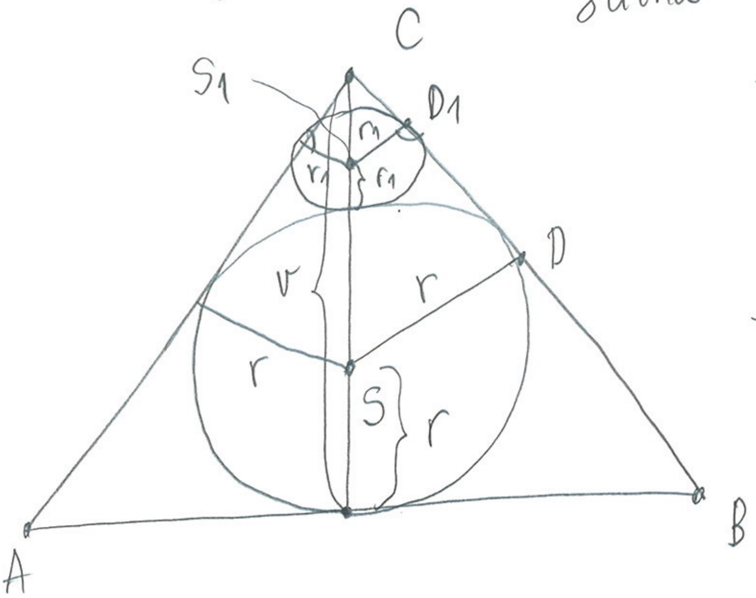
$$\Rightarrow \overline{MP} = \overline{PB} \cdot \frac{b}{a} = \frac{\sqrt{34}}{2} \cdot \frac{3}{5} = \frac{3\sqrt{34}}{10}\ cm$$

$$P_1 = \frac{\overline{MP} \cdot \overline{PB}}{2} = \frac{3\sqrt{34}}{10} \cdot \frac{\sqrt{34}}{2} \cdot \frac{1}{2} = \frac{51}{20}\ cm^2$$

$$P_2 = \frac{15}{2} - \frac{51}{20} = \frac{150-51}{20} = \frac{99}{20}\ cm^2 = 4{,}95\ c$$

(7) (Primjer 15)

Visina na osnovicu jednakokračnog trokuta iznosi 8 cm, a polumjer trokutu upisane kružnice je 2 cm. Koliki je polumjer kružnice koja dira upisanu kružnicu i krakove trokuta?



sličnost trokuta:

$$\frac{\overline{CS_1}}{r_1} = \frac{\overline{CS}}{r}$$

$r = 2$ cm

$v = 8$ cm

$r_1 = ?$

$$\overline{CS} = v - r = 8 - 2 = 6 \text{ cm}$$

$$\overline{CS_1} = -(r_1 + 2r) + v = -(r_1 + 2\cdot 2) + 8 = 8 - r_1 - 4 = 4 - r_1$$

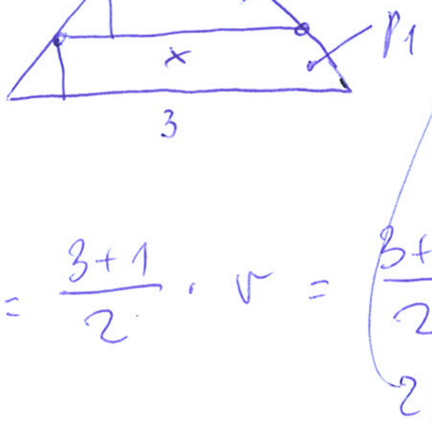
$$\frac{\overline{CS_1}}{r_1} = \frac{6}{2} = 3 \Rightarrow \overline{CS_1} = 3r_1$$

$$3r_1 = 4 - r_1$$

$$4r_1 = 4$$

$$\boxed{r_1 = 1 \text{ cm}}$$

(8) (Pr 17)



$$2\left(\frac{v_1}{v_2} + 1\right) = x + 1$$

$$\frac{3+x}{x+1} = \frac{v_1}{v_2}$$

$$P_1 = P_2$$

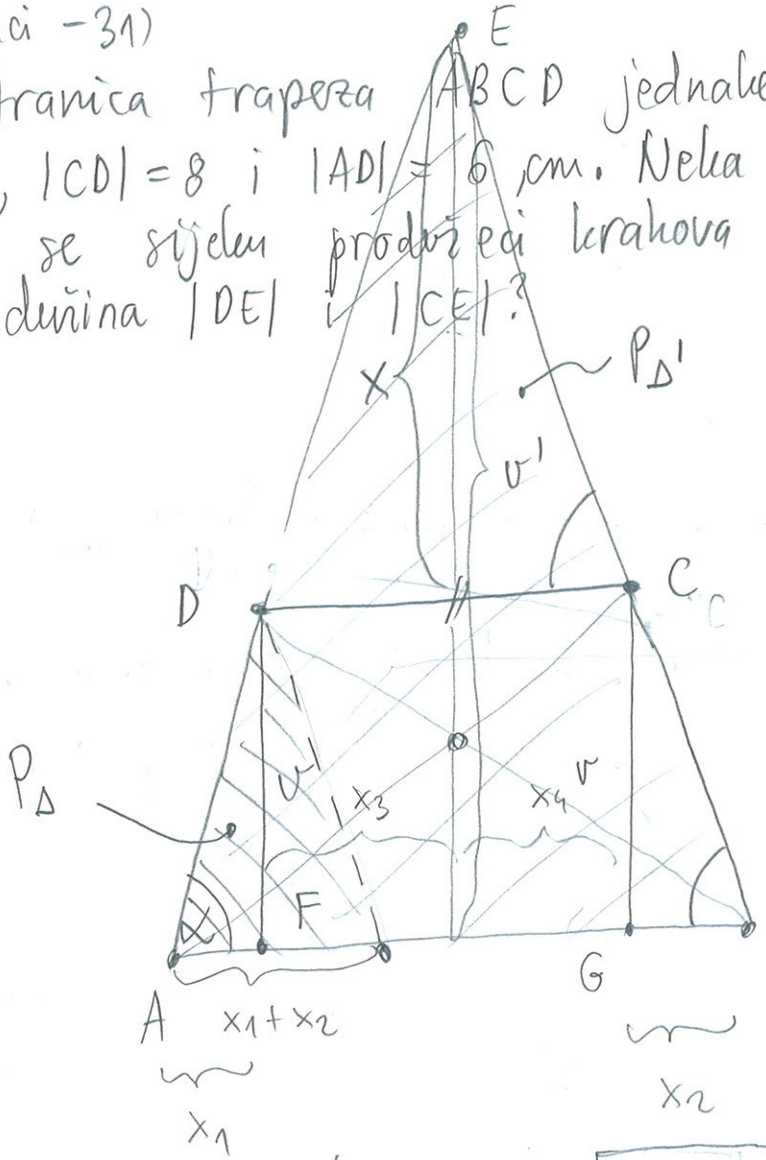
$$\left.\begin{aligned} P_1 &= \frac{3+x}{2}\cdot v_1 \\ P_2 &= \frac{x+1}{2}\cdot v_2 \end{aligned}\right\} (:)$$

$$P_{uk} = \frac{3+1}{2}\cdot v = \frac{3+1}{2}\cdot(v_1 + v_2) = 2\cdot P_2 = \cancel{2}\cdot\frac{x+1}{\cancel{2}}\cdot v_2$$

$$2(v_1 + v_2) = (x+1)\cdot v_2 \quad / : v_2$$

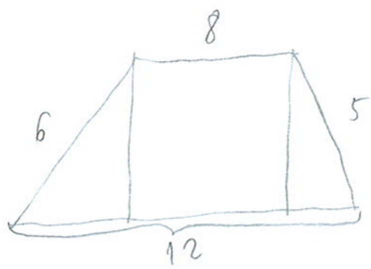
(8) (zadaci – 31)

Duljine stranica trapeza ABCD jednake su $|AB| = 12$, $|BC| = 5$, $|CD| = 8$ i $|AD| = 6$ cm. Neka je E točka u kojoj se sijeku produžeci krahova trapeza. Kolike su duljine dužina $|DE|$ i $|CE|$?



$|AB| = 12$ cm
$|BC| = 5$ cm
$|CD| = 8$ cm
$|AD| = 6$ cm

$|DE| = ?$, $|CE| = ?$



po Euklidovom poučku: $v = \sqrt{(12 - x_1)\cdot x_1} = \sqrt{(12 - x_2)\cdot x_2}$ /(1

$12 = 8 + x_1 + x_2 \quad (2) \Rightarrow x_1 + x_2 = 4$

$v^2 = 12x_1 - x_1^2$

$v = \sqrt{36 - x_1^2} \quad (3)$ $/^2$

$v^2 = 36 - x_1^2$

$12x_1 - \cancel{x_1^2} = -\cancel{x_1^2} + 36$

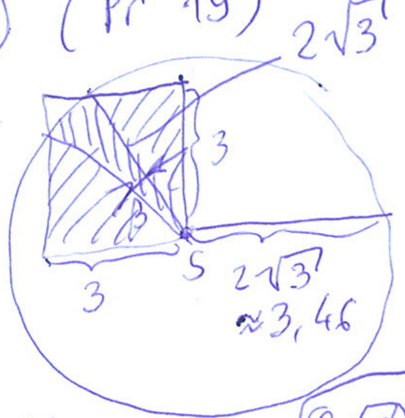
$12x_1 - 36 = 0 \quad /:12$

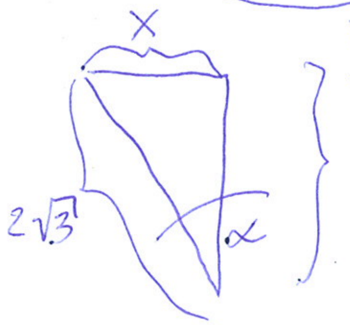
$x_1 = 3$ cm

iz (2): $x_2 = 4 - x_1 = 4 - 3 = 1$ cm

$P_{\Delta} = \frac{3\cdot\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

(9) (Pr 19)





$x = \sqrt{(2\sqrt{3})^2 - 3^2}$

$x = \sqrt{12 - 9} = \sqrt{3}$ cm

$\beta = 30°$

$\alpha = 30°$

$\cos\alpha = \frac{3}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{2\cdot 3} = \frac{\sqrt{3}}{2}$

$P_{K1} = \frac{r^2\pi\alpha}{360°}$

$P_{KC} = \frac{(2\sqrt{3})^2\pi\cdot 3}{360°}$

$P_{K1} = \frac{\cancel{4\cdot 3}\cdot\pi\cdot}{\cancel{12}}$

$$\frac{\overline{AD}}{v} = \frac{\overline{AE}}{v'}$$

$$\boxed{P_{\Delta'} = \frac{(x_1 + x_2) \cdot v}{2} = \frac{\cancel{4}^{2} \cdot 3\sqrt{3}}{\cancel{x_1}} = 6\sqrt{3}\ cm^2}$$

$$\frac{3\sqrt{3}}{2}$$

$$\boxed{P_{ok} = P_{kr} + 2P_{\Delta} = \pi + 2 \cdot \frac{3\sqrt{3}}{2} = \pi + 3\sqrt{3}}$$

$$\frac{P_{\Delta'}}{P_{\Delta'}} = \frac{v'}{v}$$

$$P_{\Delta'} = \frac{\overline{AD} \cdot x'}{2} = 4x$$

$$x_3 + x_4 = 8$$

$$\frac{x_3}{x_4} = \frac{3}{1} = 3 \Rightarrow x_3 = 3x_4$$

$$4x_4 = 8$$

$$\boxed{x_4 = 2\ cm}$$

$$\boxed{x_3 = 6\ cm}$$

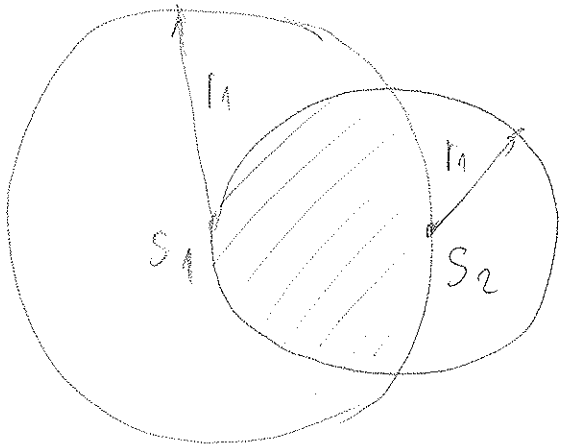
$$\sin\alpha = \frac{x_1}{AD} = \frac{3}{6} = \frac{1}{2}$$

$$\frac{x_3}{\overline{DE}} = \sin\alpha \Rightarrow \boxed{\overline{DE} = \frac{x_3}{\sin\alpha} = \frac{6}{\frac{1}{2}} = 12\ cm}$$

analogno:

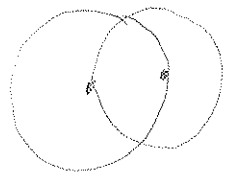
$$\boxed{\overline{CE} = \frac{x_4}{\sin\beta} = \frac{2}{\frac{1}{5}} = 10\ cm}$$

⑨ (zadaci - 44)
Središte jedne kružnice pripada drugoj, a središte druge prvoj kružnici. Ako su krugovi omeđeni tim kružnicama sukladni, kolika je površina njihovog zajedničkog dijela?



krugovi - sukladni (=)

$r_1 = r_2$



DZ

zadaci - 16, 23, 33, 38, 47

ispit 1 - 8, 9, 10

ispit 2 - 7, 8, 10

ispit 3 - 6, 7

ispit 4 - 5, 8, 10