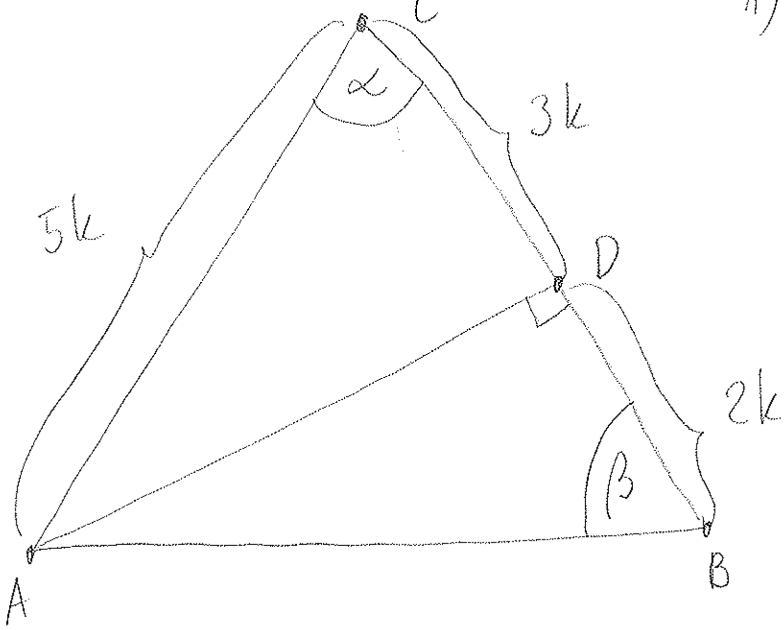
13. TRIGONOMETRIJA PRAVOKUTNOG TROKUTA

(1) Visina na krak jednakokračnog trokuta dijeli krak na dva dijela čije su duljine u omjeru 2 : 3. Koliki su kutevi ovog trokuta?
(Pr 3)

1) predpostavka $\beta > \alpha$

$$\frac{\overline{CD}}{\overline{BD}} = \frac{3}{2}$$



$$\frac{\overline{CD}}{\overline{AC}} = \frac{3\cancel{k}}{5\cancel{k}} = \cos\alpha$$

$$\cos\alpha = 0{,}6$$

$$\Rightarrow \alpha = \arccos 0{,}6$$

$$\alpha = 53{,}13^\circ$$

$$\boxed{\alpha = 53^\circ 8'}$$

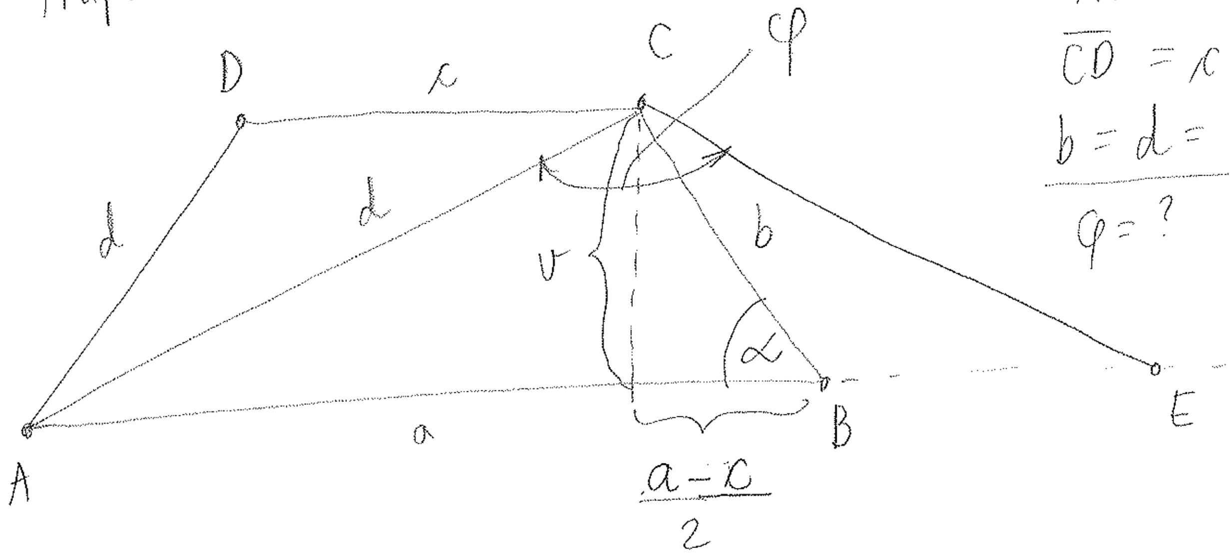
jednakokračni trokut: $\alpha + 2\beta = 180^\circ$

$$\Rightarrow \beta = \frac{180^\circ - \alpha}{2} = \frac{180^\circ - 53{,}13^\circ}{2}$$

$$\boxed{\beta = 63{,}44^\circ = 63^\circ 26'}$$

2) predpostavka: $\alpha > \beta$ $\Rightarrow$ $\boxed{\begin{array}{l} \alpha = 66^\circ 25' \\ \beta = 56^\circ 47' \end{array}}$

② (Primjer 5)
Osnovice jednakokračnog trapeza duge su 11 cm i 3 cm, duljina kraka je 7 cm. Koliki kut zatvaraju dijagonale trapeza?



$\overline{AB} = a = 11 \text{ cm}$

$\overline{CD} = c = 3 \text{ cm}$

$b = d = 7 \text{ cm}$

$\varphi = ?$

$$\frac{\frac{a-c}{2}}{b} = \cos\alpha$$

$$\cos\alpha = \frac{\frac{11-3}{2}}{7} = \frac{4}{7} \Rightarrow \alpha = \arccos\left(\frac{4}{7}\right) = 55^\circ 9'$$

$$v = b\sin\alpha = 7 \cdot \sin 55,15^\circ \approx 5,7 \text{ cm}$$

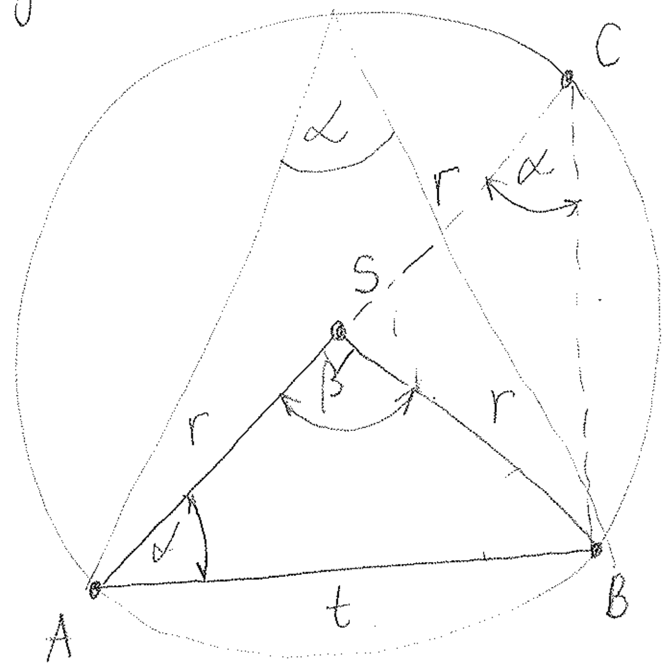
$$\frac{\left(a - \frac{a-c}{2}\right)}{v} = \operatorname{tg}\left(\frac{\varphi}{2}\right)$$

$$\operatorname{tg}\left(\frac{\varphi}{2}\right) = \frac{11 - \frac{11-3}{2}}{5,7} = \frac{7}{5,7} \approx 1,23$$

$$\Rightarrow \frac{\varphi}{2} = \operatorname{arctg}(1,23) = 50,845^\circ$$

$$\boxed{\varphi = 2 \cdot 50,845^\circ = 101,69^\circ = 101^\circ 41'}$$

③ (Primjer 6) (?)
Koliki je središnji kut nad tetivom kružnice, ako je duljina te tetive jednaka $\frac{3}{4}$ duljine promjera kružnice?



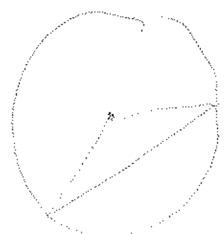
$\beta = ?$

$\underset{\sim}{t} = \overline{AB} = \frac{3}{4}\underbrace{(2r)}_{\text{promjer kružnice}} = \frac{3}{2}r$

$\sin\alpha = \frac{\overline{AB}}{\overline{AC}} = \frac{t}{2r} = \frac{\frac{3}{2}r}{2r}$

$\sin\alpha = \frac{3}{4} \Rightarrow \boxed{\alpha = 48°35'}$

obodni kut kružnice

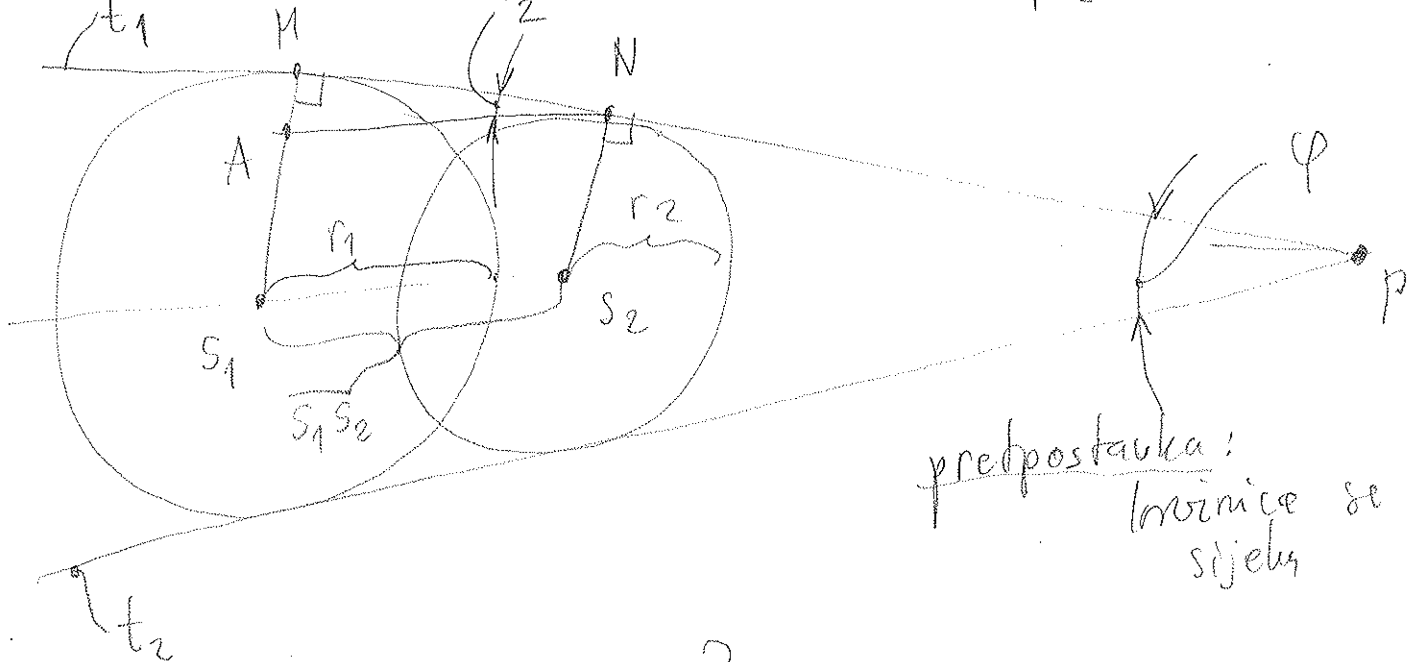


$\beta = 2\alpha$

$\boxed{\beta = 2 \cdot 48{,}59° = 97°10'}$

④ (Primjer 8)
Zajedničke vanjske tangente dviju kružnica polumjera 10 cm i 6 cm sijeku se pod kutem $\varphi = 42°45'$. Kolika je udaljenost središta ovih dviju kružnica?

$\overline{S_1S_2} = ?$



predpostavka! kružnice se sijeku

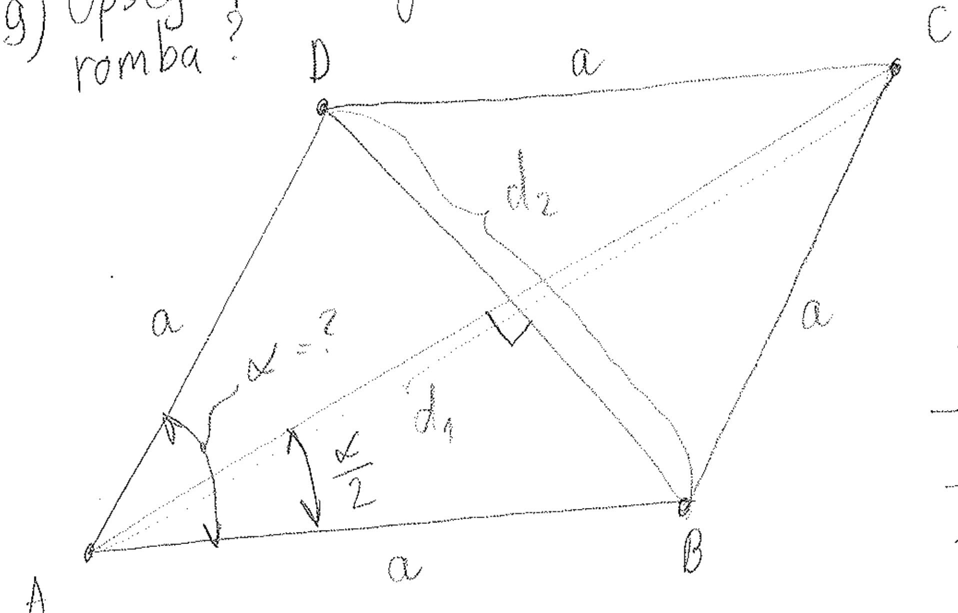
$\overline{S_1S_2} = \overline{AN}$

$\overline{AM} = r_2 - r_1 = 10 - 6 = 4$ cm

$$\frac{\overline{AM}}{\overline{AN}} = \sin\left(\frac{\varphi}{2}\right) \Rightarrow \overline{AN} = \frac{\overline{AM}}{\sin\left(\frac{\varphi}{2}\right)} = \frac{4}{\sin\left(\frac{42,75^\circ}{2}\right)}$$

$$\boxed{\overline{AN} \approx 10,97 \text{ cm}}$$

(5) Zbroj duljina dijagonala romba jednak je 28 cm.
(Pr 9) Opseg romba jednak je 40 cm. Koliki je šiljasti kut romba?



$d_1 + d_2 = 28 \Rightarrow d_1 = 28 - d_2$

$4a = 40 \Rightarrow a = 10$ cm

$$\frac{\frac{d_2}{\cancel{2}_1}}{\frac{d_1}{\cancel{2}_1}} = tg\left(\frac{\alpha}{2}\right)$$

$$tg\left(\frac{\alpha}{2}\right) = \frac{d_2}{d_1}$$

$$\left(\frac{d_2}{2}\right)^2 + \left(\frac{d_1}{2}\right)^2 = a^2 \quad / \cdot 4$$

$$d_1^2 + d_2^2 = 400$$

$$(28 - d_2)^2 + d_2^2 = 400$$

$$784 - 56d_2 + d_2^2 + d_2^2 = 400$$

$$2d_2^2 - 56d_2 + 384 = 0$$

$$d_2^2 - 28d_2 + 192 = 0$$

$$d_{2_{1,2}} = \frac{28 \pm \sqrt{(-28)^2 - 4 \cdot 192}}{2}$$

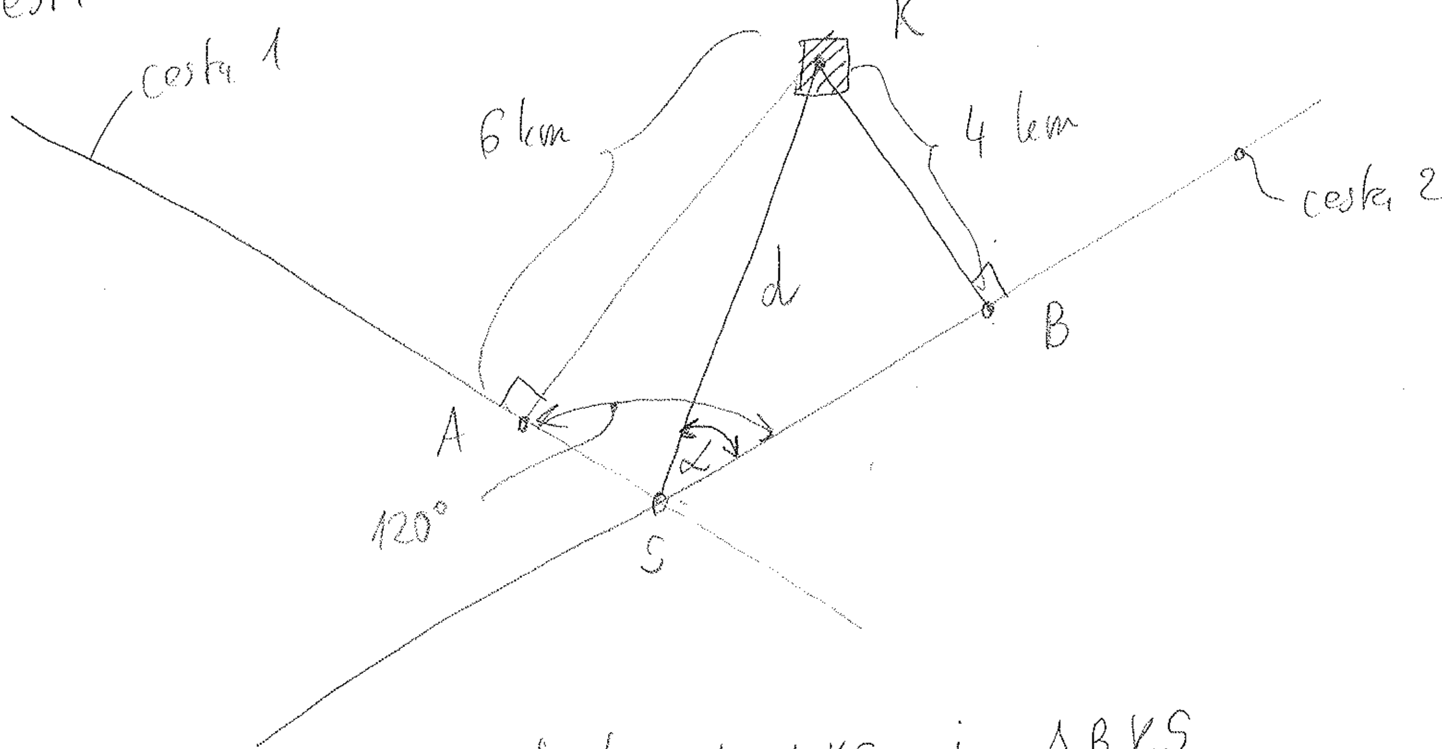
$$d_{2_{1,2}} = \frac{28 \pm 4}{2} \begin{cases} \nearrow (d_2)_1 = 12 \text{ cm} \\ \searrow (d_2)_2 = 16 \text{ cm} \end{cases}$$

Možemo uzeti da je $d_1 = 16$ cm, $d_2 = 12$ cm.

$$\operatorname{tg}\left(\frac{\alpha}{2}\right) = \frac{12}{16} = 0{,}75 \Rightarrow \frac{\alpha}{2} = 36{,}87^\circ$$

$$\boxed{\alpha = 73{,}74^\circ = 73^\circ 44'}$$

⑥ (Primjer 12)
Dvije ravne ceste sijeku se pod kutem od 120°. Neka je kuća od jedne ceste udaljena 6 km, a od druge 4 km. Kolika je udaljenost te kuće od križanja dviju cesta?



dva pravokutna trokuta ΔAKS i ΔBKS

$$\left.\begin{aligned} \frac{4}{d} &= \sin\alpha \\ \frac{6}{d} &= \sin(120^\circ - \alpha) \end{aligned}\right\} \quad \frac{4}{\sin\alpha} = \frac{6}{\sin(120^\circ - \alpha)}$$

$$\sin(120^\circ - \alpha) = \underbrace{\sin 120^\circ}_{\frac{\sqrt{3}}{2}} \cos\alpha - \sin\alpha \underbrace{\cos 120^\circ}_{\left(-\frac{1}{2}\right)}$$

primjena adicijske formule

$$= \frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha$$

$$\frac{4}{\sin\alpha} = \frac{6}{\frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha}$$

$$6\sin\alpha = 2\sqrt{3}\cos\alpha + 2\sin\alpha \quad /:2$$

$$\sqrt{3}\cos\alpha - 2\sin\alpha = 0 \quad /:\cos\alpha$$

$$\sqrt{3} - 2\,\mathrm{tg}\,\alpha = 0$$

$$\mathrm{tg}\,\alpha = \frac{\sqrt{3}}{2} \Rightarrow \alpha = 40,893^\circ$$

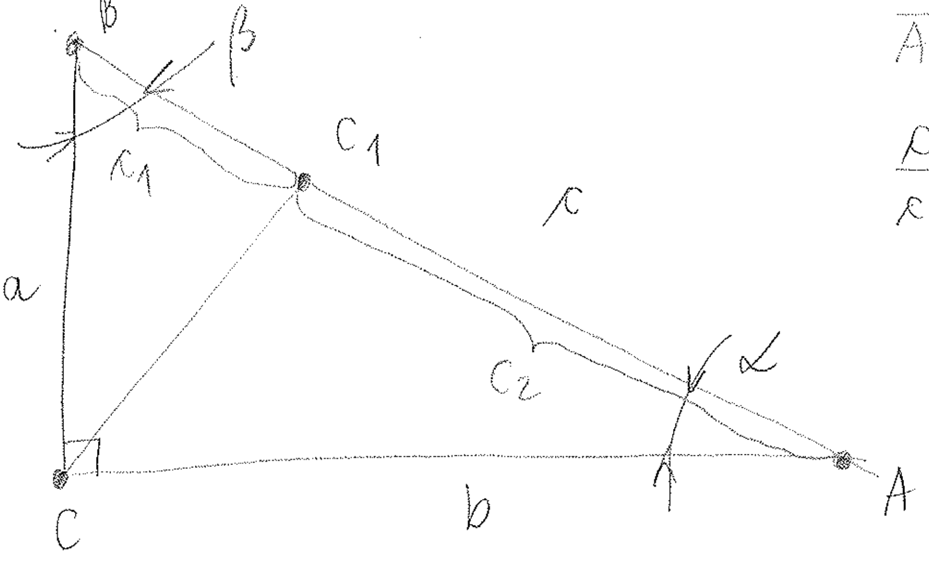
$$\boxed{d = \frac{4}{\sin 40,893^\circ} \approx 6,11 \text{ km}}$$

⑦ (zadaci – 5)
Nožište visine na hipotenuzu pravokutnog trokuta dijeli hipotenuzu na dijelove čije su duljine u omjeru 2 : 5. Koliki su kutevi trokuta?

$\overline{BC_1} = c_1$

$\overline{AC_1} = c_2$

$$\frac{c_2}{c_1} = \frac{5}{2}$$



$r_1 = 2k$

$r_2 = 5k$

$v = \sqrt{5k \cdot 2k} = k\sqrt{10}$ eulerov poučak

$$\frac{v}{r_2} = \operatorname{tg}\alpha = \frac{k\sqrt{10}}{5k} = \frac{\sqrt{10}}{5} \Rightarrow \alpha = \operatorname{arctg}\left(\frac{\sqrt{10}}{5}\right)$$

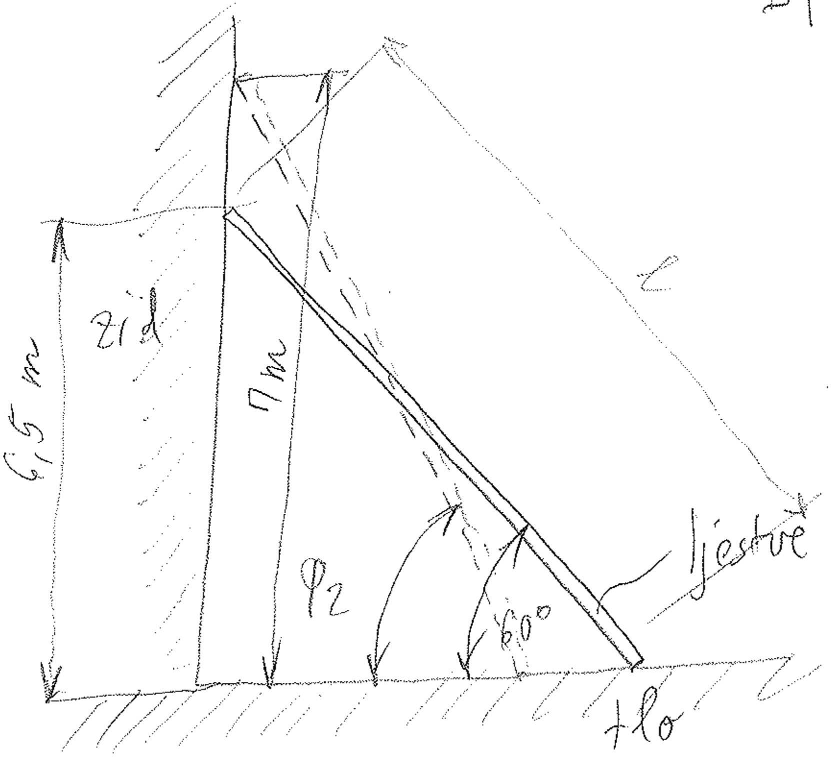
$$\boxed{\alpha = 32,312 = 32^\circ 18'}$$

$\beta = 90^\circ - \alpha = 57,688^\circ$

$$\boxed{\beta = 57^\circ 41'}$$

(8) (ispit 2-4)
Postave li se ljestve prema zidu tako da s tlom zatvaraju kut od 60°, one dosežu do visine od 6,5 m. Ako želimo da dosegnu visinu od 7 metara, za koliko se mora povećati prikloni kut?

$\Delta\varphi = \varphi_2 - \varphi_1$



$$\frac{6,5}{l} = \sin 60^\circ$$

$$l = \frac{6,5}{\sin 60^\circ} = 6,5 \cdot \frac{2}{\sqrt{3}}$$

$$l = \frac{13\sqrt{3}}{3}\ m$$

$$\sin\varphi_2 = 7 \cdot \frac{3}{13\sqrt{13}}$$

$\Rightarrow \varphi_2 = 68,851^\circ$

$\Delta\varphi = 68,851 - 60 = 8,851^\circ$

$$\boxed{\Delta\varphi = 8^\circ 51'}$$ A

⑨ (ispit 4 - 7) (0)
Duljine osnovica jednakokračnog trapeza jednake su 7 cm i 3 cm, a šiljasti kut tog trapeza jednake je 72°. Površina tog trapeza iznosi?

DZ

zadaci - 7, 10, 14
ispit 1 - 5, 7, 9
ispit 2 - 2, 7, 9, 10
ispit 3 - 8, 9
ispit 4 - 3, 6, 10

14. POUČCI O TROKUTU I PRIMJENE TRIGONOMETRIJE

nakon ① *

① (Primjer 4)
Duljine stranica trokuta zadovoljavaju uvjet $\frac{(b+c)^2-a^2}{bc}=1$

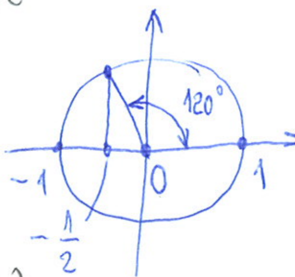
Koliki je kut α?

$$\frac{(b+c)^2-a^2}{bc}=1 \quad / \cdot bc$$

$$(b+c)^2-a^2=bc$$

$$a^2=(b+c)^2-bc=b^2+2bc+c^2-bc$$

$$a^2=b^2+bc+c^2 \quad (1)$$



prema kosinusovom poučku vrijedi:

$$a^2=b^2+c^2-2bc\cos\alpha \quad (2) \quad / \cdot (-1)$$

①* (1) + (2): $bc+2bc\cos\alpha=0 \Rightarrow \cos\alpha=-\frac{1}{2}$

$$\boxed{\alpha=\frac{2\pi}{3}=120^\circ}$$

(1a) (Pr 3)
$O_\Delta=55$ cm
$\alpha=46^\circ\,27'=46{,}45^\circ$
$\beta=63^\circ\,15'=63{,}25^\circ$

$P_\Delta=?$

$P=\frac{c^2\sin\alpha\sin\beta}{2\sin\gamma}$

$O=a+b+c$

$a=c\frac{\sin\alpha}{\sin\gamma}$

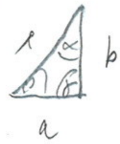
$b=c\frac{\sin\beta}{\sin\gamma}$

$\alpha+\beta+\gamma=180^\circ \Rightarrow \gamma=180^\circ-(\alpha+\beta)$

② (Primjer 5)
Duljine stranica trokuta tri su uzastopna cijela broja, najveći je kut trokuta dva puta veći od najmanjeg.
Koliki je opseg tog trokuta?

$\gamma=180^\circ-(46{,}45^\circ+63{,}25^\circ)$

$\gamma=70{,}3^\circ$



$n-1,\ n,\ n+1$ ... stranice trokuta

$\gamma=2\alpha$

poučak o sinusima:

$$\frac{a}{\sin\alpha}=\frac{c}{\sin\gamma}$$

$$\frac{n-1}{\sin\alpha}=\frac{n+1}{\sin 2\alpha}$$

$P=\frac{20{,}23^2\sin 46{,}45^\circ\sin 63{,}25^\circ}{2\sin 70{,}3^\circ}$

$55=c\left(\frac{\sin\alpha}{\sin\gamma}+\frac{\sin\beta}{\sin\gamma}+1\right)$

$\boxed{P=140{,}67\ \text{cm}^2}$

$c=\frac{55}{\frac{\sin 46{,}45^\circ}{\sin 70{,}3^\circ}+\frac{\sin 63{,}25^\circ}{\sin 70{,}3^\circ}+1}$

$c=20{,}23$ cm

$$\frac{n+1}{n-1} = \frac{\sin 2\alpha}{\sin\alpha} = \frac{2\cancel{\sin\alpha}\cos\alpha}{\cancel{\sin\alpha}} = 2\cos\alpha$$

$$\cos\alpha = \frac{1}{2}\,\frac{n+1}{n-1}$$

– primjena poučka o kosinusima:

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$(n-1)^2 = n^2 + (n+1)^2 - 2n(n+1)\cos\alpha$$

$$(n-1)^2 = n^2 + (n+1)^2 - \cancel{2}n(n+1)\frac{1}{\cancel{2}}\,\frac{n+1}{n-1} \quad / \cdot (n-1)$$

$$(n-1)^3 = n^2(n-1) + (n+1)^2(n-1) - n(n+1)^2$$

$$n^3 - 3n^2 + 3n - 1 = n^3 - n^2 + (n^2+2n+1)(n-1) - n(n^2+2n+1)$$

$$\cancel{n^3} - 3n^2 + 3n - \cancel{1} = \cancel{n^3} - n^2 + \cancel{n^3} + \cancel{2n^2} + \cancel{n} - n^2 - 2n - \cancel{1} -$$

$$- \cancel{n^3} - \cancel{2n^2} - \cancel{n}$$

$$-n^2 + 5n = 0 \quad / : n$$

$$n = 5$$

$$\boxed{\begin{aligned} a &= 4\text{ cm} \\ b &= 5\text{ cm} \\ c &= 6\text{ cm} \end{aligned}} \Bigg\} \quad \boxed{O = a + b + c = 4 + 5 + 6 = 15\text{ cm}}$$

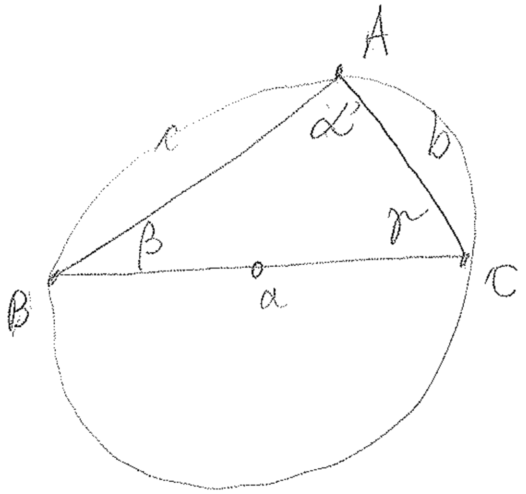
③ (Primjer 7)
U trokutu $\triangle ABC$ za duljine stranica vrijedi $a-b=5\,cm$, $c=7\,cm$. Duljina polumjera trokutu opisane kružnice je $R=\frac{7\sqrt{3}}{3}$. Kolika je duljina stranice a?

$a-b=5\,cm$ (1)

$c=7\,cm$ (2)

$R=\frac{7\sqrt{3}}{3}\,cm$

$a=?$



- prema poučku o sinusima i polumjeru opisane kružnice:

$$\frac{c}{\sin\gamma}=2R \Rightarrow \sin\gamma=\frac{c}{2R}=\frac{\cancel{7}^{1}}{2\cdot\frac{\cancel{7}\sqrt{3}}{3}}$$

$$\sin\gamma=\frac{3}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{3}}{2}$$

$$\Rightarrow \gamma=60° \text{ ili } 120° \quad (3)$$

I) II)

poučak o kosinusu:

$$c^2=a^2+b^2-2ab\cos\gamma \quad (4)$$

I) (1), (2), (3) → (4)

$$49=a^2+(a-5)^2-\cancel{2}^{1}a(a-5)\underbrace{\cos 60°}_{=\frac{1}{\cancel{2}_1}}$$

$$49=\cancel{a^2}+a^2-10a+25-\cancel{a^2}+5a=0$$

$$a^2-5a-24=0 \rightarrow a_1=8\,cm$$

$$a_2=\cancel{-3}$$

$$a_{1,2}=\frac{5\pm\sqrt{25+96}}{2\cdot 1}$$

11) $49 = a^2 + (a-5)^2 - 2a(a-5)\underbrace{\cos 120^\circ}_{= -\frac{1}{2}}$

$$49 = a^2 + a^2 - 10a + 25 - (2a^2 + 10a)\cdot\left(-\frac{1}{2}\right)$$

$$49 = a^2 + a^2 - 10a + 25 + a^2 + 5a$$

$$3a^2 - 15a - 24 = 0$$

$$a_{3,4} = \frac{15 \pm \overbrace{\sqrt{(-15)^2 - 4\cdot 3\cdot(-24)}}^{17,7}}{2\cdot 3}$$

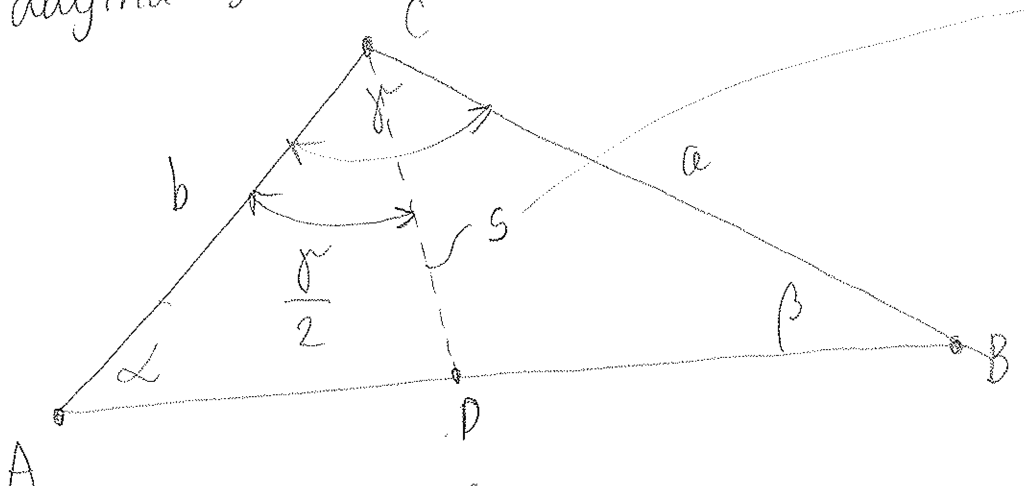
$\rightarrow a_3 \approx 6,3$ cm

$\rightarrow a_4 \approx \cancel{-1,3}$ cm

za $\gamma = 60^\circ$	$\longrightarrow$	$a = 8$ cm
za $\gamma = 120^\circ$	$\longrightarrow$	$a = 6,3$ cm

4 (Primjer 9)
U trokutu ΔABC zadan je $\gamma = 120^\circ$, a za duljine stranica a i b vrijedi jednakost $ab = a + b$. Kolika je duljina simetrale kuta γ?

$\rightarrow s = ?$



$$\underbrace{P_{\Delta ABC}}_{\frac{1}{2}ab\sin\gamma} = \underbrace{P_{\Delta ACD}}_{\frac{1}{2}sb\sin\left(\frac{\gamma}{2}\right)} + \underbrace{P_{\Delta BCD}}_{\frac{1}{2}sa\sin\left(\frac{\gamma}{2}\right)} \quad /\cdot 2$$

$$\underbrace{ab\sin 120^\circ}_{\frac{\sqrt{3}}{2}} = sb\underbrace{\sin 60^\circ}_{\frac{\sqrt{3}}{2}} + sa\underbrace{\sin 60^\circ}_{\frac{\sqrt{3}}{2}} \quad / \cdot \frac{2}{\sqrt{3}}$$

$$\left.\begin{array}{ll} ab = sb + sa & (1) \\ ab = a + b & (2) \end{array}\right\} (=)$$

$$s(a+b) = a+b \Rightarrow \boxed{s = 1 \text{ cm}}$$

(5) (zadaci - 4)
Ako su duljine stranica trokuta u omjeru 2:5:6, koliki je najveći kut trokuta?

$a : b : c = 2 : 5 : 6$

najveći kut nasuprot najveće stranice $\rightarrow c \rightarrow \gamma = ?$

$$\left.\begin{array}{l} a = 2k \\ b = 5k \\ c = 6k \end{array}\right\}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$$36k^2 = 4k^2 + 25k^2 - 2 \cdot 2k \cdot 5k \cdot \cos\gamma \quad / : k^2$$

$$36 = 29 - 20 \cdot \cos\gamma$$

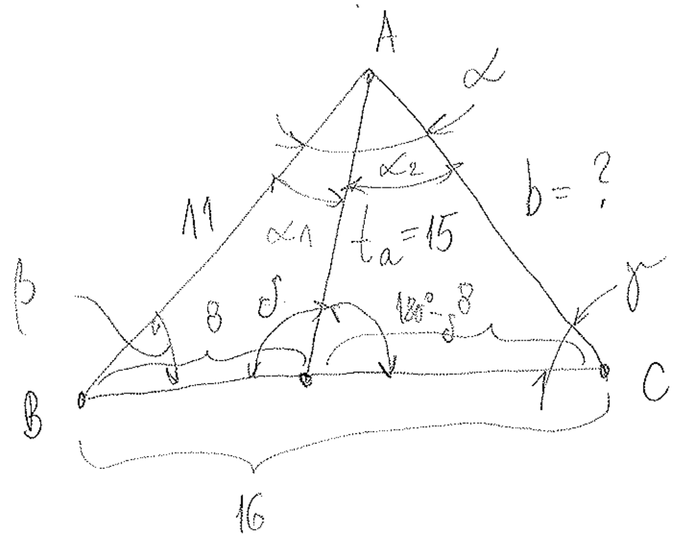
$$7 = -20\cos\gamma \Rightarrow \cos\gamma = -\frac{7}{20}$$

$$\gamma = \arccos\left(-\frac{7}{20}\right) = 110{,}487^\circ$$

$$\boxed{\gamma = 110^\circ 29' 13''}$$

⑥ (zadaci 12)
Duljina težišnice povučene iz vrha A trokuta ΔABC jednaka je $t_a = 15$ cm, a još su zadane duljine stranica, $a = 16$ cm i $c = 11$ cm. Izračunajte duljinu stranice b.

$$\underbrace{c^2}_{=11^2} = \overset{\frac{a}{2}}{8^2} + \overset{t_a}{15^2} - 2 \cdot 8 \cdot 15 \cdot \cos\delta$$



$$121 = 64 + 225 - 240 \cdot \cos\delta$$

$$-240 \cos\delta = -168$$

$$\cos\delta = 0.7$$

$$\Rightarrow \underline{\delta = 45{,}573°}$$

$$180° - \delta = 180 - 45{,}573°$$

$$\underline{180° - \delta = 134{,}43°}$$

$$\frac{t_a}{\sin\beta} = \frac{c}{\sin\delta} \Rightarrow \sin\beta = t_a \cdot \frac{\sin\delta}{c} = \frac{15}{11} \cdot \sin 45{,}573°$$

$$\Rightarrow \underline{\beta = 76{,}863°}$$

$$\alpha_1 = 180° - \beta - \delta = 134{,}43° - 76{,}863°$$

$$\underline{\alpha_1 = 57{,}567°}$$

$$\cancel{\frac{t_a}{\sin\gamma} = \frac{\frac{a}{2}}{\sin\alpha_2}}$$

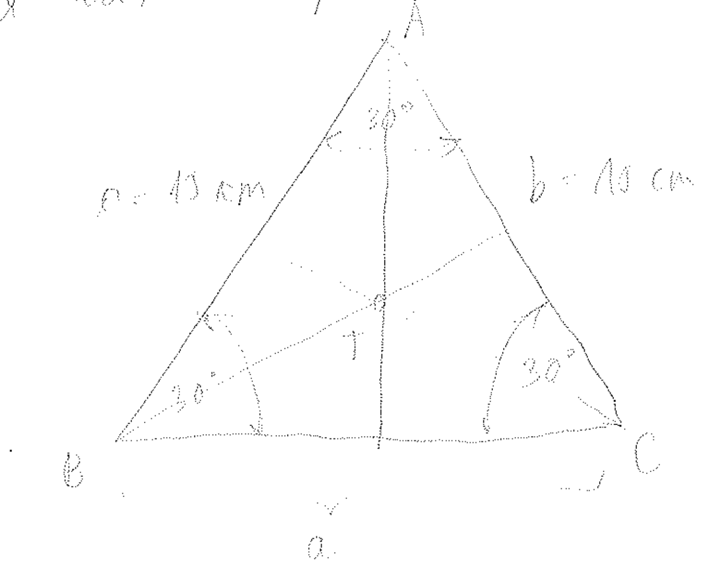
$$b^2 = t_a^2 + \left(\frac{a}{2}\right)^2 - \cancel{2}^{1} \cdot t_a \cdot \frac{a}{\cancel{2}_{1}} \cdot \cos(180° - \delta)$$

$$\boxed{b = \sqrt{15^2 + 8^2 - 15 \cdot 8 \cdot \cos 134{,}43°} = 19{,}3 \text{ cm}}$$

DZ

zadaci – 5, 13, 14, 16
ispit 1 – 7, 8, 9, 10
ispit 2 – 6, 8, 9

(1) (zadaci 14)
Kolika je duljina simetrale kuta uz osnovicu jednakokračnog trokuta ako je duljina kraka jednaka 15 cm, a kut nasuprot osnovici iznosi 30°?

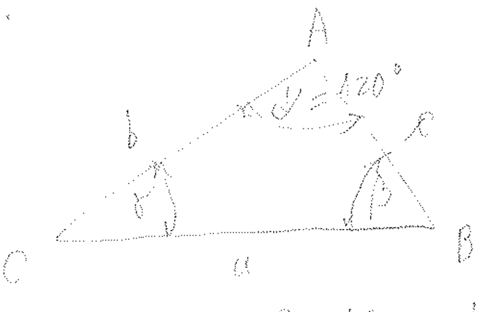


$s = v$

$\frac{v}{c} = \cos\left(\frac{30°}{2}\right) = \cos 15°$

$v = 15 \cdot \cos 15° = 14,5$ cm

(2) (ispit 1 – 6)
U trokutu ΔABC duljine su stranica a, b i c, te vrijede jednakosti $a - b = b - c = 5$ cm. Jedan kut tog trokuta iznosi 120°. Duljina stranice nasuprot tom kutu jednaka je:



Kosinusov poučak:

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

$a = \sqrt{b^2 + c^2 - 2bc\cos\alpha}$

$a - b = 5 \Rightarrow a = 5 + b$

$b - c = 5 \Rightarrow c = b - 5$

$a = \sqrt{b^2 + (b-5)^2 - 2b(b-5)\cos 120°}$

$(5+b) = \sqrt{b^2 + (b-5)^2 - 2b(b-5)\left(-\frac{1}{2}\right)} \quad /^2$

$(b+5)^2 = b^2 + (b-5)^2 + b^2 - 5b$

$\cancel{b^2} + 10b + \cancel{25} = \cancel{25} + b^2 - 10b + \cancel{b^2} + b^2 - 5b$

$2b^2 - 25b = 0$

$b(2b - 25) = 0$

$b_1 = 0$ odbacujemo

$b = \frac{25}{2}$

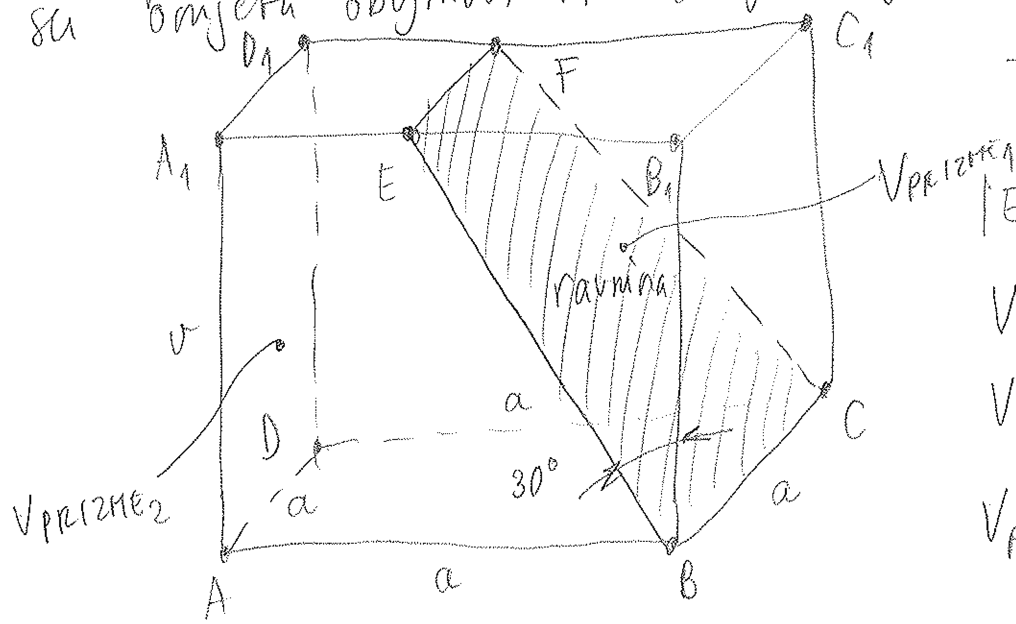
$a = \frac{25}{2} + 5$

$a = 17,5$ cm

(B)

15. POLIEDRI I ROTACIJSKA TIJELA

① (Primjer 3)
Ravnina položena osnovnim bridom kocke pod kutem od 30° prema osnovici dijeli kocku na dva dijela. U kojem su omjeru obujmovi tih dvaju dijelova?



$$\frac{EB_1}{a} = tg\,30° = \frac{\sqrt{3}}{3}$$

$$|EB_1| = a\frac{\sqrt{3}}{3}$$

$$V_K = a^3$$

$$V_{PRIZME\,1} = a \cdot \frac{a\sqrt{3}}{3} \cdot \frac{1}{2} \cdot a$$

$$V_{PRIZME\,1} = \frac{a^3\sqrt{3}}{6}$$

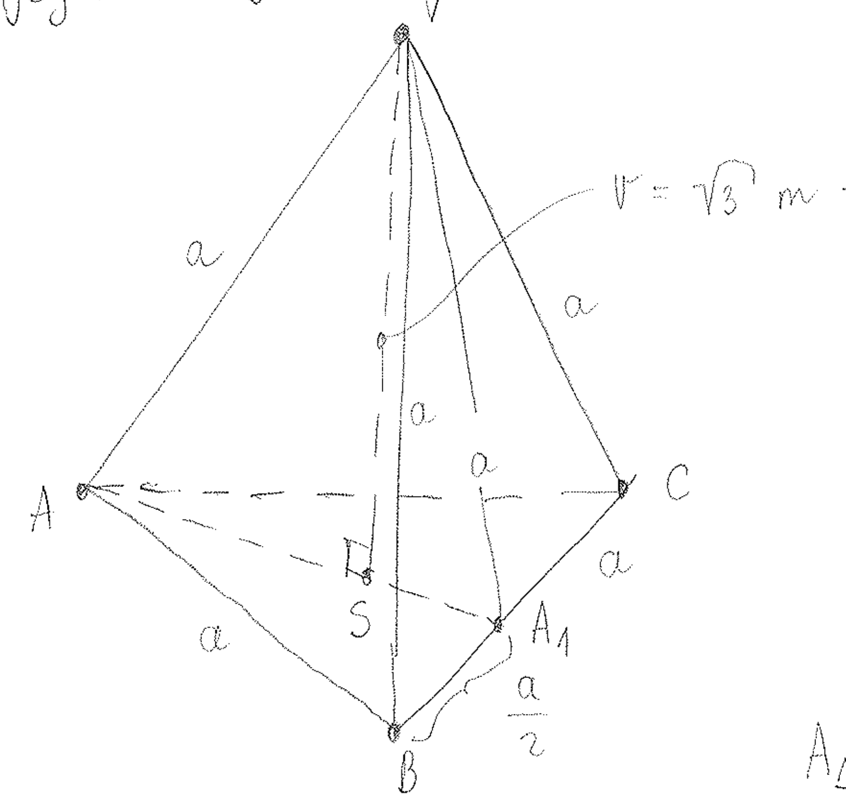
$$V_{PRIZME\,2} = a^3 - \frac{\sqrt{3}}{6}a^3 = \frac{6-\sqrt{3}}{6}a^3$$

$$\boxed{\frac{V_{PRIZME\,2}}{V_{PRIZME\,1}}} = \frac{\frac{6-\sqrt{3}}{6}\cancel{a^3}}{\frac{\sqrt{3}}{6}\cancel{a^3}} = \frac{6-\sqrt{3}}{\cancel{6}} \cdot \frac{\cancel{6}}{\sqrt{3}} = \frac{6-\sqrt{3}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

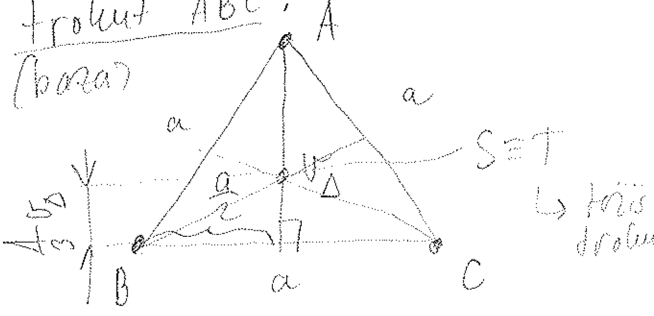
omjeri obujama $= \dfrac{6\sqrt{3}-3}{3} = \boxed{2\sqrt{3}-1}$

② (Primjer 4)

Visina pravilnog tetraedra, trostrane piramide kojoj su svi bridovi iste duljine, jednaka je $\sqrt{3}$ m. Koliki je njegov obujam?



trokut ABC:
(baza)



$$V = \frac{1}{3} B \cdot v$$

$$B = A_{\Delta ABC} = \frac{1}{2} a \cdot v_\Delta$$

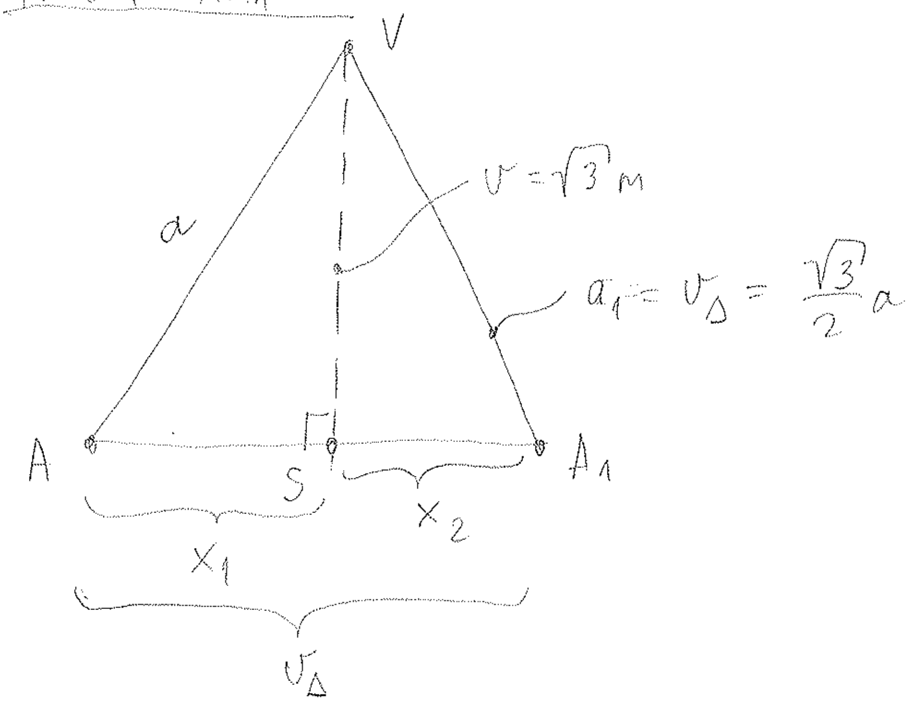
$$v_\Delta = \sqrt{a^2 - \frac{a^2}{4}} = \sqrt{\frac{3}{4} a^2} = \frac{\sqrt{3}}{2} a$$

$$A_{\Delta ABC} = \frac{1}{2} a \cdot \frac{\sqrt{3}}{2} a = \frac{\sqrt{3}}{4} a^2$$

$$V = \frac{1}{3} \cdot \frac{\sqrt{3}}{4} a^2 \cdot \sqrt{3}$$

$$V = \frac{1}{4} a^2$$

trokut AA_1V:



Kako je S ujedno i težište trokuta:

$$x_1 = \frac{2}{3} v_\Delta = \frac{\cancel{2}}{3} \cdot \frac{\sqrt{3}}{\cancel{2}} a$$

$$x_1 = \frac{\sqrt{3}}{3} a$$

Pitagorin poučak:

$$a^2 = 3 + \frac{1}{3} a^2$$

$$\frac{2}{3} a^2 = 3 \quad / \cdot \frac{3}{2}$$

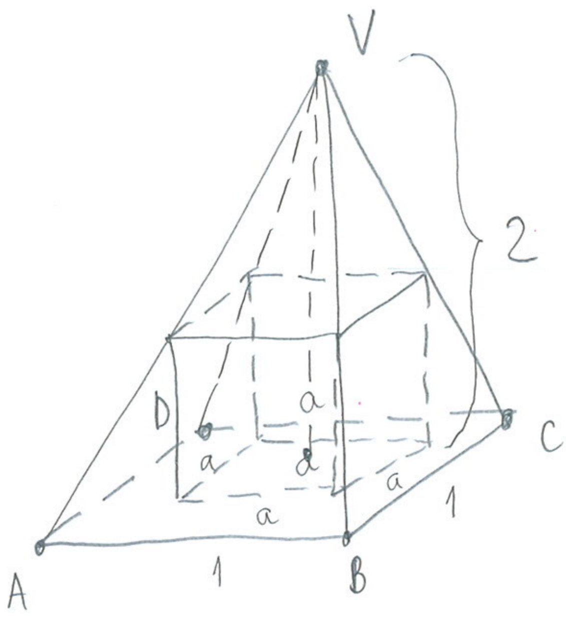
$$a^2 = \frac{9}{2}$$

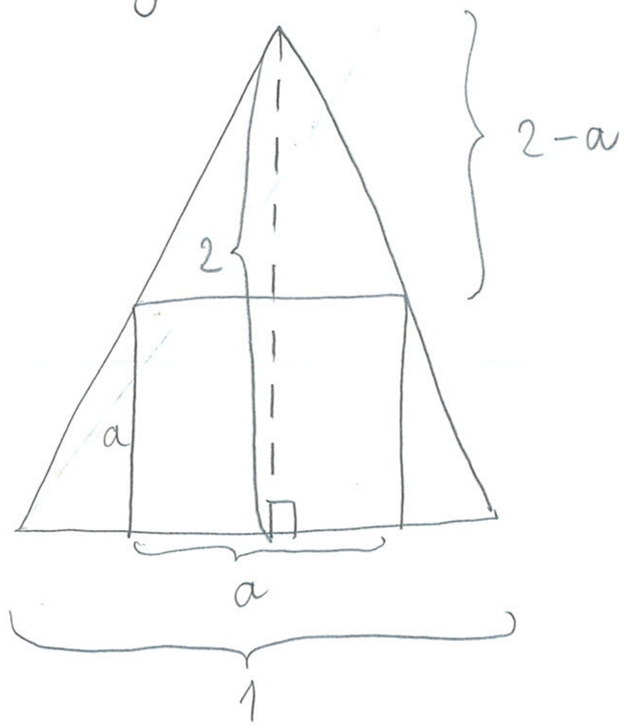
$$\Rightarrow a = \frac{3\sqrt{2}}{2}\ m$$

$$\boxed{V = \frac{1}{4} \cdot \frac{9}{2} = \frac{9}{8}\ m^3}$$

(3) (Primjer 5)

U pravilnu četverostranu piramidu visine 2 m i osnovnog brida 1 m upisana je kocka kojoj je jedna strana na bazi piramide, a vrhovi njoj suprotne strane su na bočnim bridovima piramide. Kolika je duljina brida kocke?





Prema sličnosti trokuta:

$$\frac{1}{2} = \frac{a}{2-a}$$

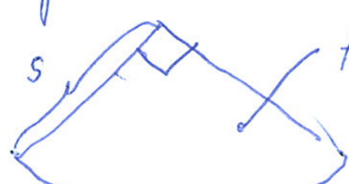
$$2a = 2 - a$$

$$3a = 2 \Rightarrow \boxed{a = \frac{2}{3}\ m}$$

(Primjer 7)

$v = \sqrt{s^2 - r^2}$

plašt stošca



$A = 100\pi\ cm^2$

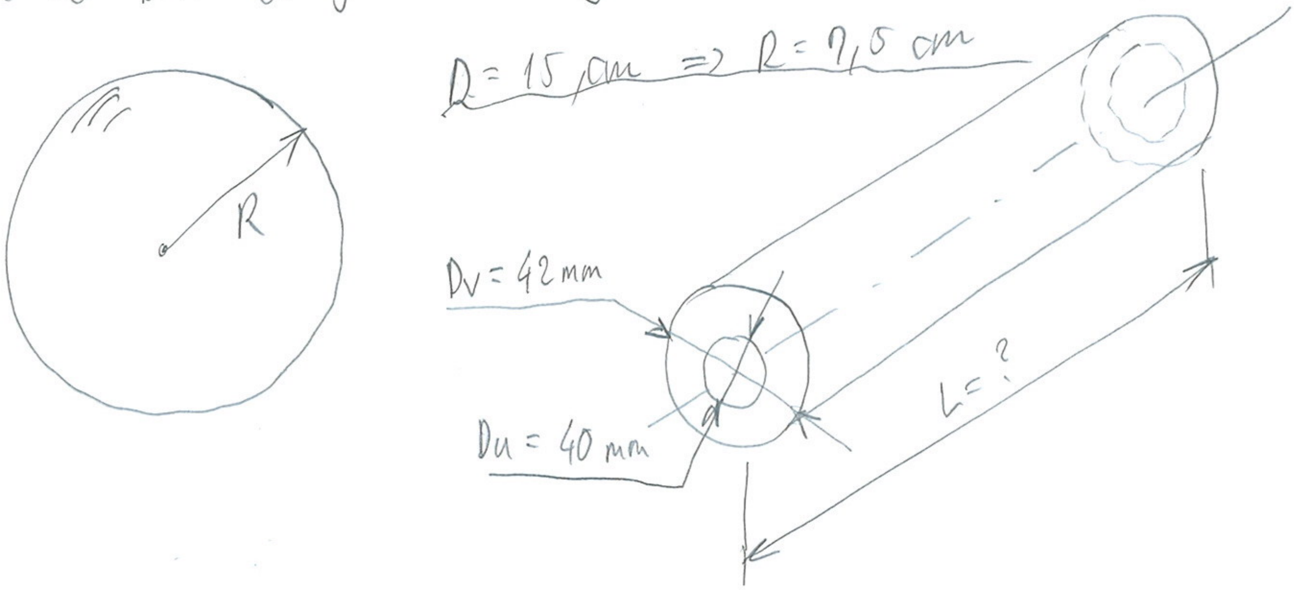
kružni isječak

opseg osnovke stošca $\quad 2r\pi = \frac{s\pi\alpha}{180^\circ}$ $\quad P_{ki} = \frac{s^2\pi\alpha}{180^\circ} = A \Rightarrow s = 20\ cm$

$\Rightarrow r = 5\ cm$

④ (Primjer 8)

Metalnu kuglu promjera 15 cm valja pretopiti u valjkastu cijev vanjskog promjera 42 mm i unutarnjeg 40 mm. Kolika će biti duljina te cijevi?



$$V_K = V_{cijevi}$$

$$V_K = \frac{4}{3} R^3 \pi = \frac{4}{3} \cdot 7{,}5^3 \pi = 562{,}5\,\pi \ \text{cm}^3$$

$$V_{cijevi} = \left(\frac{D_v^2 - D_u^2}{4}\right) \pi L = \left(\frac{4{,}2^2 - 4^2}{4}\right) \pi L = 0{,}41 \cdot \pi L \ [\text{cm}^3]$$

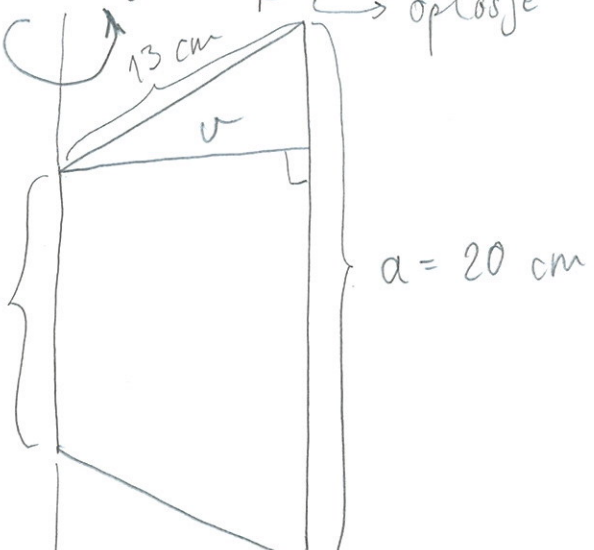
$$562{,}5\,\pi = 0{,}41 \cdot \pi L \Rightarrow \boxed{L = 1371{,}951 \text{ cm} = 13{,}72 \text{ m}}$$

Prije (Primjer 11)

⑤ (Primjer 12)

Jednakokračan trapez s osnovicama duljina 20 cm i 10 cm, te krakovima duljine 13 cm vrti se oko manje osnovice. Izračunajmo površinu i obujam rotacijskog tijela.

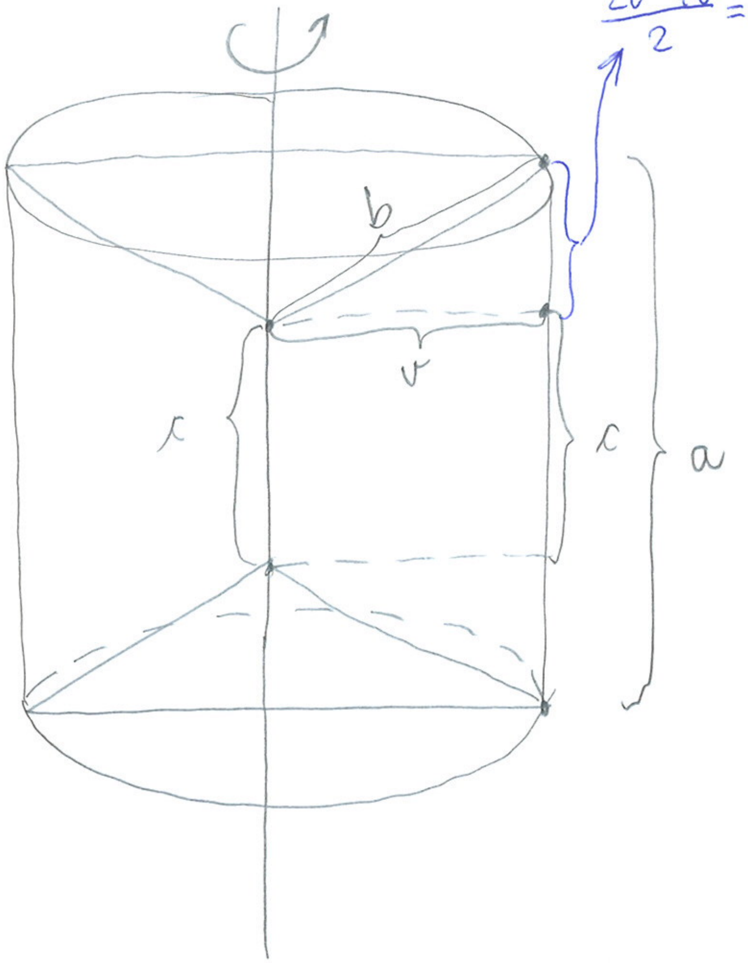
površinu → oplošje



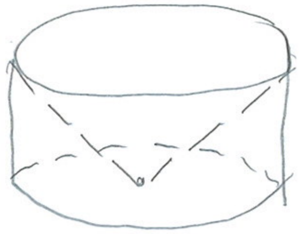
$\frac{20-10}{2} = 5$

$v = \sqrt{\cancel{6}^2 - 5^2}$

$v = \sqrt{13^2 - 5^2} = 12\ cm$



Oplošje tijela: $O = 2O_1 + O_2$



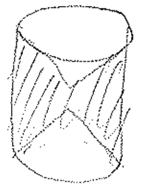
$O_1 = v\pi \cdot b = 12 \cdot \pi \cdot 13$

$2O_1 = 2 \cdot 12 \cdot 13 \cdot \pi = 312\pi\ cm^2$

$O_2 = 2v\pi \cdot a = 2 \cdot 12 \cdot \pi \cdot 20 = 480\pi\ cm^2$

$\boxed{O = (312 + 480)\pi = 792\pi\ cm^2}$

Volumen tijela:



$$V = 2V_1 + V_2$$

ili

$$V = V_v - 2V_s$$

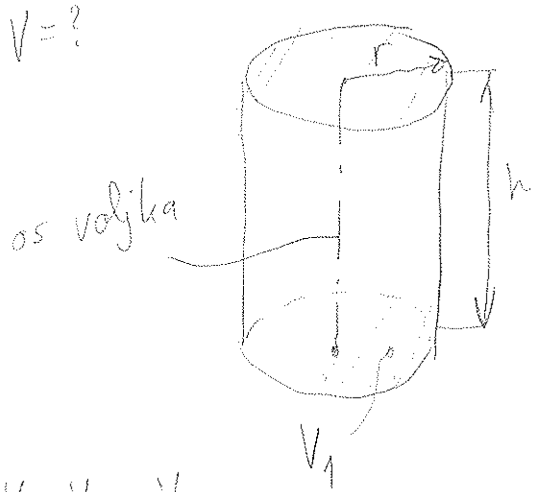
$$V_v = r^2\pi \cdot a - 2 \cdot \frac{1}{3} r^2\pi \cdot \left(\frac{a-c}{2}\right)$$

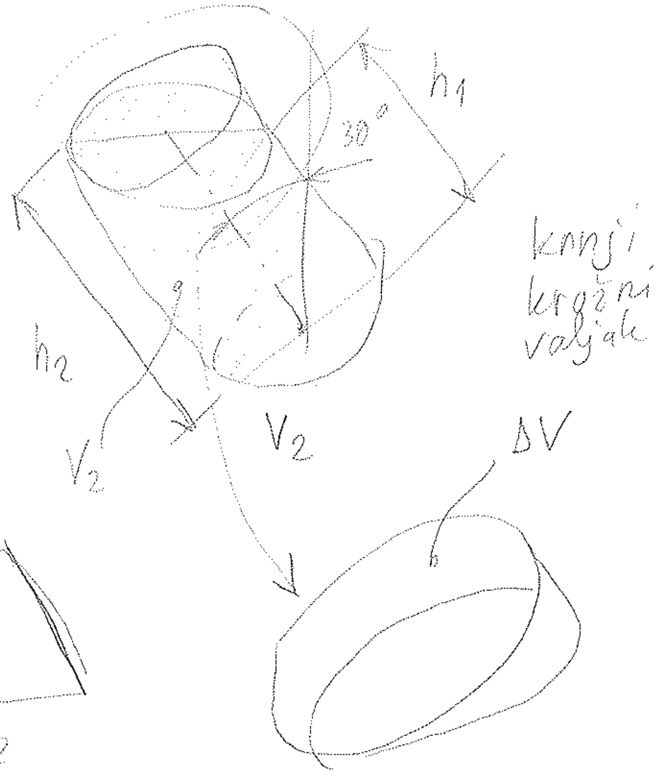
$$\boxed{V = 12^2\pi \cdot 20 - \frac{2}{3} \cdot 12^2 \cdot \pi \cdot 5 = 2880 \cdot \pi - 480\pi = 2400\pi \text{ cm}^3}$$

⑥ (zadaci – 13)

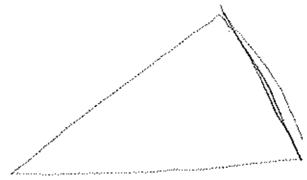
Ako je čaša oblika valjka puna vode, pa je nagnemo prema osi valjka za 30°, koliko će se vode izliti iz čaše?

$\Delta V = ?$





$$\Delta V = V_1 - V_2$$

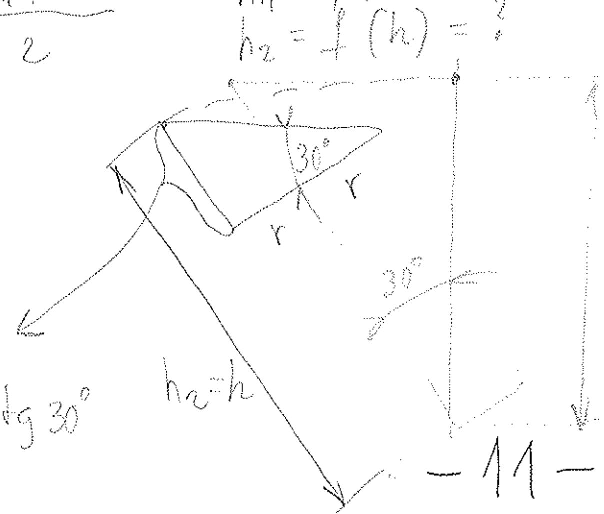


$$V_1 = r^2\pi \cdot h$$

$$V_2 = \pi r^2 \frac{h_1 + h_2}{2}$$

$h_1 = f(h) = ?$

$h_2 = f(h) = ?$



$$\frac{x}{2r} = \operatorname{tg} 30^\circ$$

$$h_2 = h$$

$$h_1 = h_2 - x$$

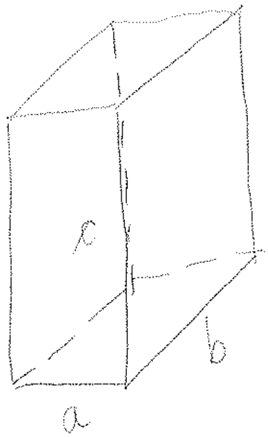
$$h_1 = h - \frac{2\sqrt{3}}{3} r$$

$x = 2r \cdot tg\,30° = 2r \cdot \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3} r$

$V_2 = r^2 \pi \left(h + h - \frac{2\sqrt{3}}{3} r\right) \cdot \frac{1}{2} = r^2 \pi \left(h - \frac{\sqrt{3}}{3} r\right)$

$\boxed{\Delta V = r^2 \pi h - \left(r^2 \pi \left(h - \frac{\sqrt{3}}{3} r\right)\right) = \frac{\sqrt{3}}{3} r}$

17 (ispit 1 – 1)
Duljine bridova kvadra u omjeru su 1 : 2 : 3. Ako je obujam kvadra jednak 162 cm², njegovo oplošje iznosi?



$a : b : c = 1 : 2 : 3 \quad (1)$

$V = 162\ cm^2$

$O = ?$

$O = 2(ab + bc + ac)$

$V = abc = 162$

iz (1):

$\frac{a}{b} = \frac{1}{2} \Rightarrow b = 2a$

$\frac{a}{c} = \frac{1}{3} \Rightarrow c = 3a$

$a \cdot 2a \cdot 3a = 162$

$6a^3 = 162$

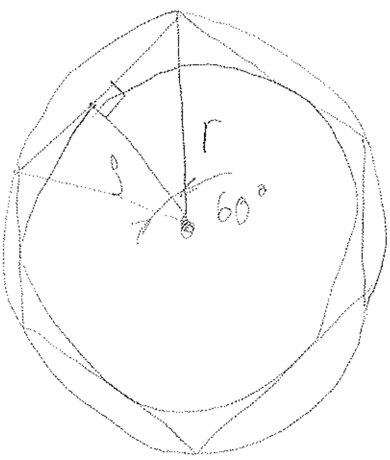
$a^3 = 27 \Rightarrow a = 3\ cm$

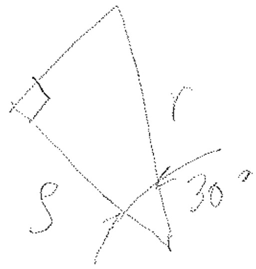
$b = 6\ cm$

$c = 9\ cm$

$\boxed{O = 2(3 \cdot 6 + 6 \cdot 9 + 3 \cdot 9) = 198\ cm^2}$ A

⑧ (ispit 1 − 7)
Pravilnoj šesterostranoj piramidi upisan je i opisan stožac tako da sva tri dijela imaju zajedničku visinu i da su im osnovke u istoj ravnini. Volumeni upisanog i opisanog stošca u omjeru su:





$\rho = f(r) = ?$

$\rho = r\cos 30° = \frac{\sqrt{3}}{2} r$

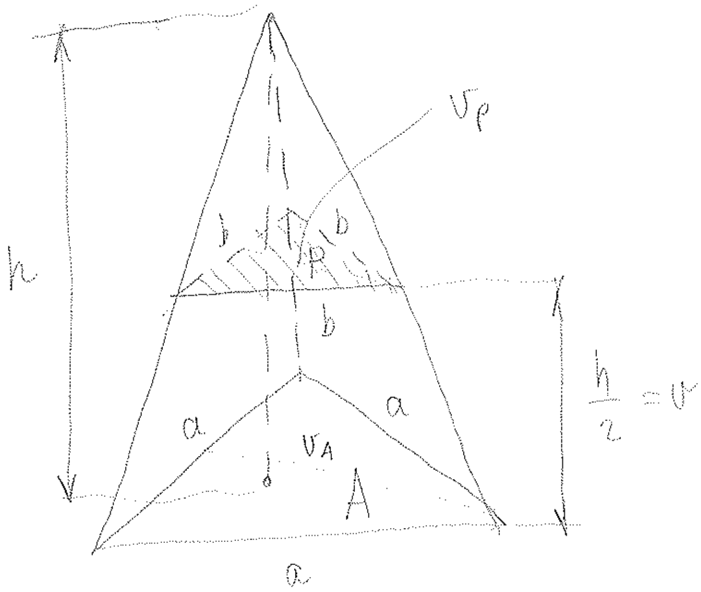
$$V_{SV} = \frac{1}{3} r^2 \pi h$$

$$V_{SU} = \frac{1}{3} \rho^2 \pi h = \frac{1}{3}\left(\frac{\sqrt{3}}{2} r\right)^2 \pi h$$

$$\boxed{\frac{V_{SU}}{V_{SV}} = \frac{\cancel{\frac{1}{3}}\left(\frac{\sqrt{3}}{2} r\right)^2 \cancel{\pi h}}{\cancel{\frac{1}{3}} r^2 \cancel{\pi h}} = \frac{3}{4}}$$ ⓒ

⑨ Piramida je presječena ravninom paralelno osnovci polovištem visine. Ako je obujam pritom dobivene krnje piramide jednak 27 cm³, onda je obujam cijele piramide? ~~koja je baza?~~ (~~trokut, kvadrat, ...~~)
(ispit 2 − 5)

DZ zadaci − 3, 6, 9, 16, 20
ispit 1 − 4, 8, 10
ispit 2 − 1, 6, 8, 9



$$V_{KP} = \frac{v}{3}\left(A + \sqrt{AP} + P\right)$$

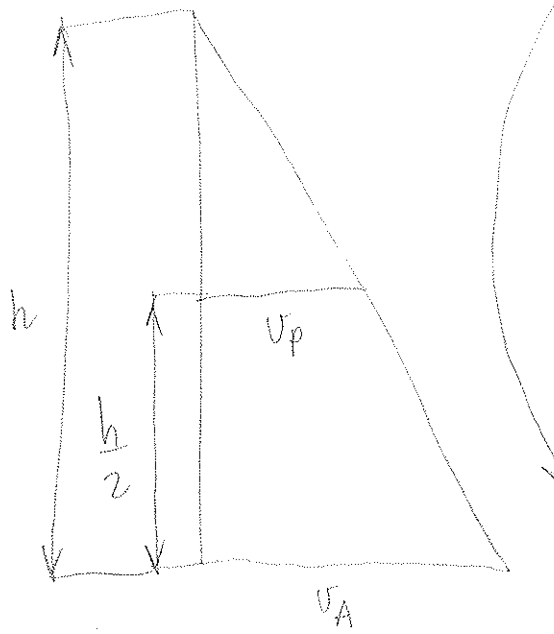
$$V_{KP} = 27\ cm^2$$

$$A = \frac{a^2\sqrt{3}}{4}$$

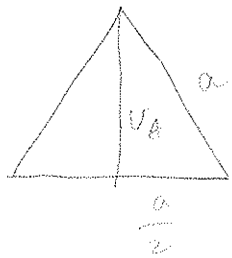
$$P = \frac{b^2\sqrt{3}}{4}$$

$$v = \frac{h}{2}$$

$$b = f(a) = ?$$



$$\frac{v_P}{\frac{\cancel{h}}{2}} = \frac{v_A}{\cancel{h}}$$



$$v_A = \sqrt{a^2 - \frac{a^2}{4}}$$

$$v_A = \sqrt{\frac{3}{4}a^2} = \frac{\sqrt{3}}{2}a$$

$$v_P = \frac{\sqrt{3}}{2}b$$

$$v_P = \frac{1}{2}v_A = \frac{\sqrt{3}}{4}a$$

$$\Rightarrow b = \frac{2\sqrt{3}}{3}v_P$$

$$\frac{v}{3}\left(A + \sqrt{AP} + P\right) = \frac{1}{3}\cdot\frac{h}{2}\cdot\left(\frac{a^2\sqrt{3}}{4} + \sqrt{\frac{a^2\sqrt{3}}{4}\cdot\frac{1}{4}a^2\frac{\sqrt{3}}{4}} + \frac{1}{4}\frac{a^2\sqrt{3}}{4}\right) = (\circ)$$

$$b = \frac{\cancel{2}^{1}\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{\cancel{4}_{2}}a = \frac{3}{6}a = \frac{1}{2}a$$

$$(\circ) = \frac{h}{6}\cdot\left(\frac{a^2\sqrt{3}}{4} + \frac{a^2\sqrt{3}}{16} + \sqrt{\sqrt{9}}\ \frac{a^2}{4}\cdot\frac{1}{2}\right)$$

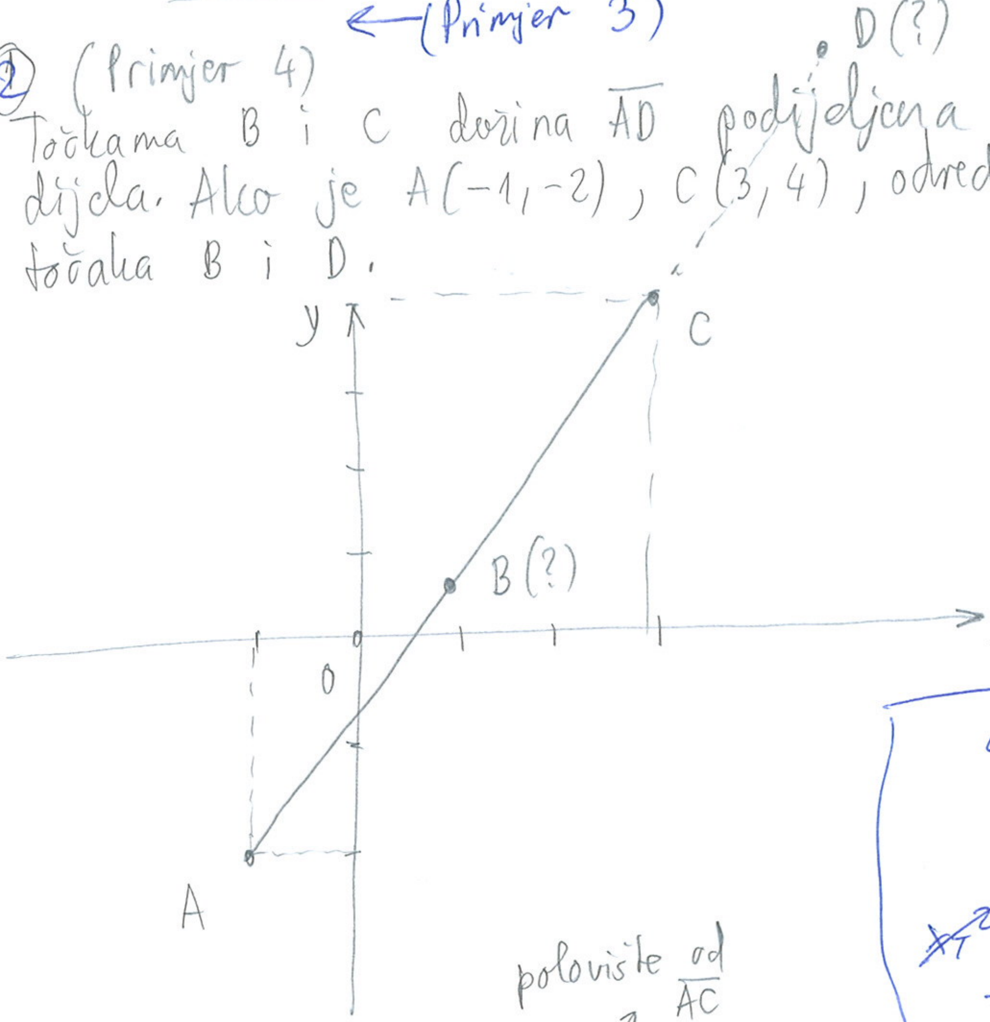
$$= \frac{h}{6}\cdot\left(\frac{4+1+2}{16}\sqrt{3}a^2\right) = \frac{7\sqrt{3}}{96}a^2h = 27$$

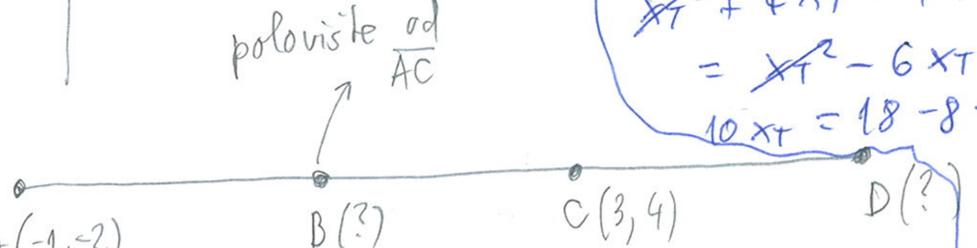
16. KOORDINATNI SUSTAV U RAVNINI

← (Primjer 3)

② (Primjer 4)

Točkama B i C dužina $\overline{AD}$ podijeljena je na tri sukladna dijela. Ako je $A(-1,-2)$, $C(3,4)$, odredimo koordinate točaka B i D.





$$4 + (x_T+2)^2 = 9 + (x_T-3)^2$$

$$\cancel{x_T^2} + 4x_T + 4 + 4 = \cancel{x_T^2} - 6x_T + 9 + 9$$

$$10x_T = 18 - 8 = 10 \Rightarrow x_T = 1$$

$T(1,0)$

$$x_B = \frac{x_A + x_C}{2} = \frac{-1+3}{2} = 1$$

$$y_B = \frac{y_A + y_C}{2} = \frac{-2+4}{2} = 1$$

$\boxed{B(1,1)}$

$$x_C = \frac{x_B + x_D}{2} \qquad 3 = \frac{1 + x_D}{2} \Rightarrow \boxed{x_D = 6 - 1 = 5}$$

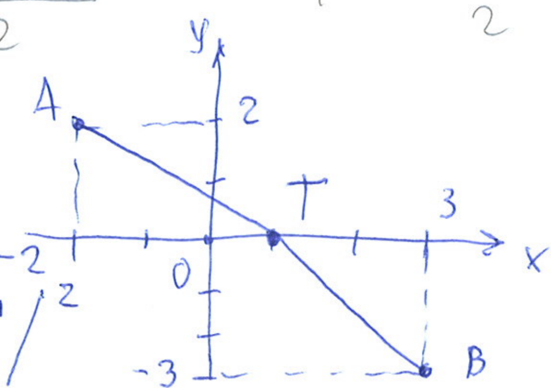
$$y_C = \frac{y_B + y_D}{2} \qquad 4 = \frac{1 + y_D}{2} \Rightarrow \boxed{y_D = 8 - 1 = 7}$$

$\boxed{D(5,7)}$

① (Pr 3) $A(-2,2)$, $B(3,-3)$

$$\sqrt{(y_T-2)^2 + (x_T+2)^2} = \sqrt{(\cancel{y_T}+3)^2 + (3+x_T)^2}$$

$= 0$



$|\overline{AT}| = |\overline{BT}|$

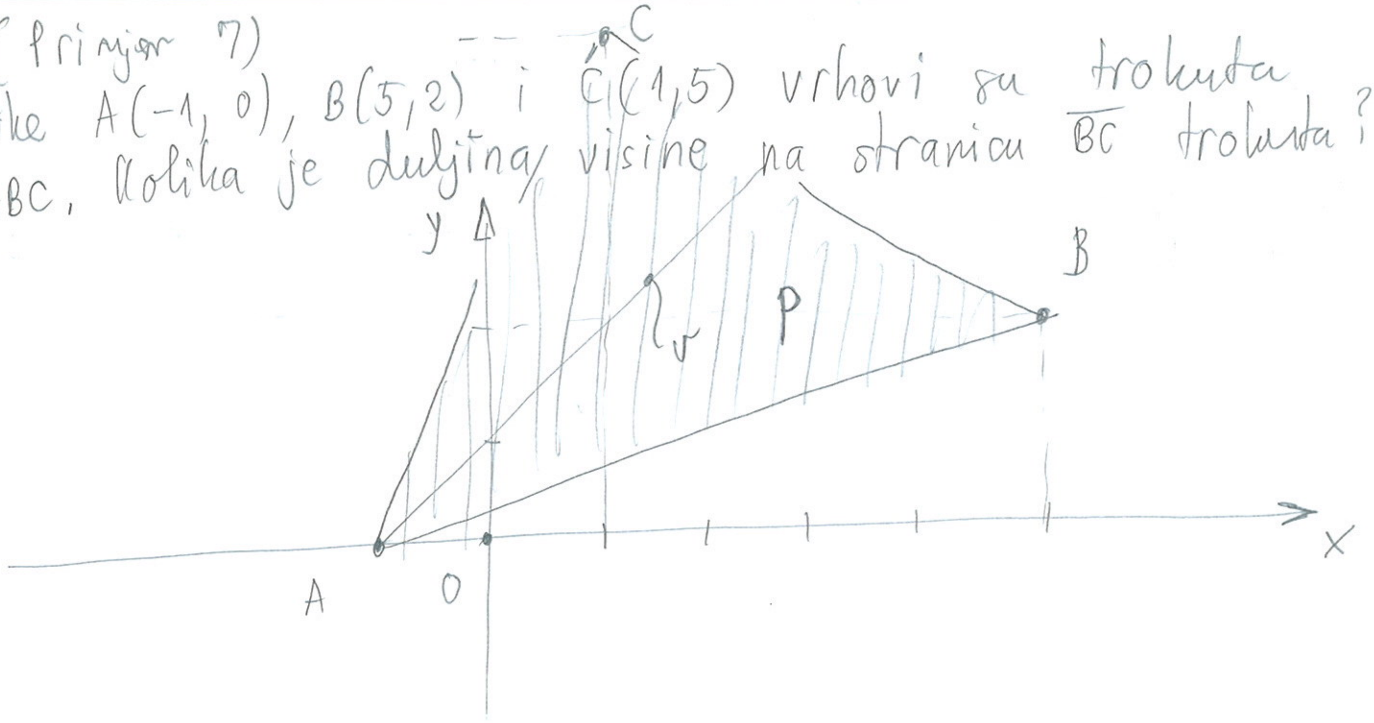
$$\sqrt{(y_T-y_A)^2 + (x_T-x_A)^2} = \sqrt{(y_B-y_T)^2 + (x_B-x_T)^2}$$

$$a^2 h = 27 \cdot \frac{96}{7\sqrt{3}} = \frac{2592\sqrt{3}}{21} = 213,785$$

$$\boxed{V_p = \frac{1}{3} B \cdot h = \frac{1}{3} A \cdot h = \frac{1}{3} \underbrace{a^2 \frac{\sqrt{3}}{4} h}_{213,785} = 30,86 \text{ cm}^2}$$ D

(4) (Primjer 7)

Točke $A(-1,0)$, $B(5,2)$ i $C(1,5)$ vrhovi su trokuta ΔABC. Kolika je duljina visine na stranicu $\overline{BC}$ trokuta?



$$P=\frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$$

$$P=\frac{1}{2}\left[-1(2-5)+5(5-0)+1(0-2)\right]$$

$$\boxed{P=\frac{1}{2}\left[3+25-2\right]=13}$$

$$P=\frac{|BC|\cdot v}{2} \Rightarrow v=\frac{2P}{|BC|}$$

$$|BC|=\sqrt{(y_C-y_B)^2+(x_C-x_B)^2}=\sqrt{(5-2)^2+(1-5)^2}=\sqrt{9+16}=5$$

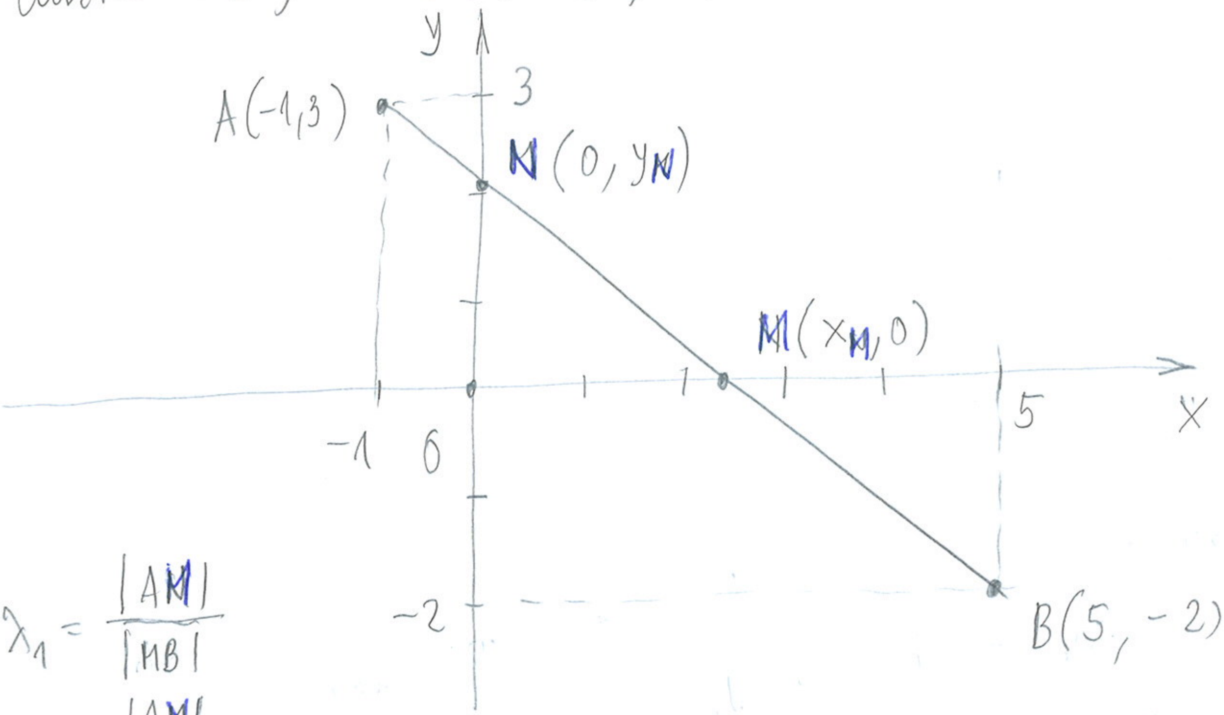
$$\boxed{v=\frac{2\cdot 13}{5}=\frac{26}{5}=5{,}2}$$

(Pr 5)

(3)

(5) (Primjer 9)

U kojim omjerima sjecišta s koordinatnim osima dijele dužinu $\overline{AB}$, $A(-1,3)$, $B(5,-2)$?



$$\lambda_1 = \frac{|AM|}{|MB|}$$

$$\lambda_2 = \frac{|AN|}{|NB|}$$

$$\overrightarrow{AM} = \lambda_1 \overrightarrow{MB}$$

$$y_M = \frac{y_A + y_B \cdot \lambda_1}{1 + \lambda_1}$$

$$0 = \frac{3 + (-2) \cdot \lambda_1}{1 + \lambda_1} \quad / \cdot (1 + \lambda_1)$$

$$3 - 2\lambda_1 = 0 \Rightarrow \boxed{\lambda_1 = \frac{3}{2}}$$

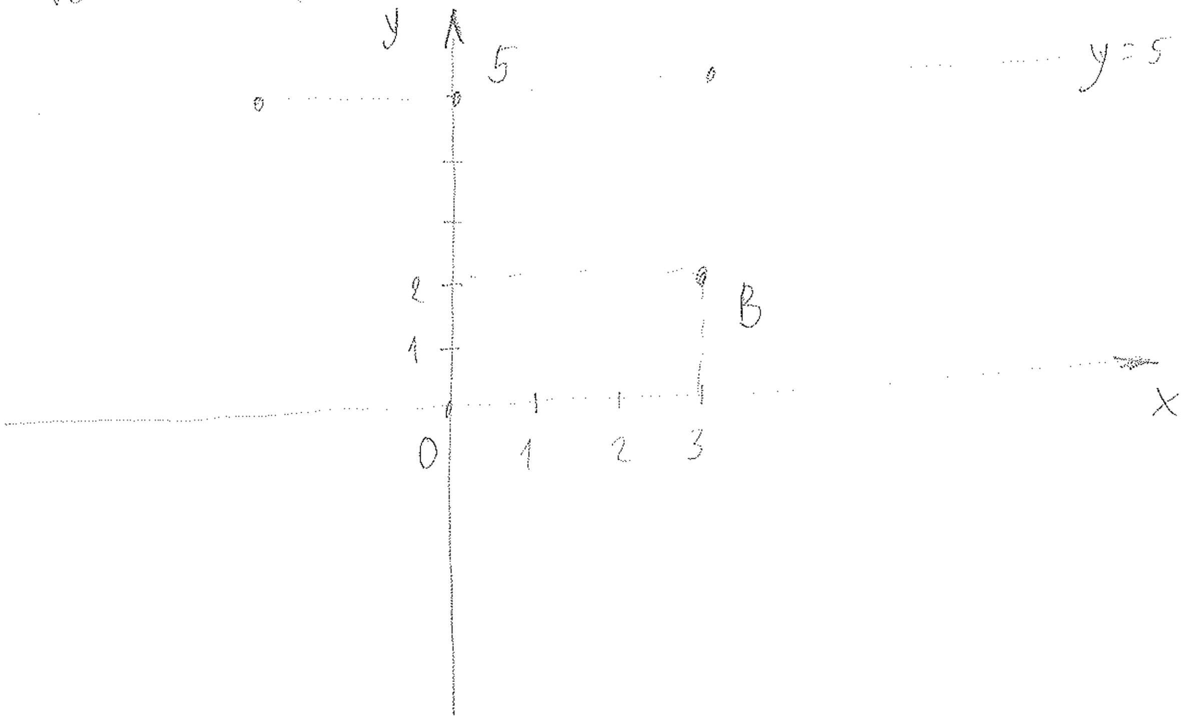
$$x_N = \frac{x_A + x_B \cdot \lambda_2}{1 + \lambda_2}$$

$$0 = \frac{-1 + 5 \cdot \lambda_2}{1 + \lambda_2} \quad / \cdot (1 + \lambda_2)$$

$$0 = -1 + 5 \cdot \lambda_2 \Rightarrow \boxed{\lambda_2 = \frac{1}{5}}$$

8 (zadaci – 4)

Točka $A(x, 5)$ jednako je udaljena od osi apcisa i od točke $B(3,2)$. Odredimo apcisu točke A.



$$|AB| = \sqrt{(y_B - y_A)^2 + (x_B - x_A)^2} = \sqrt{(2-5)^2 + (3-x)^2}$$

$$|Ax| = 5$$

$$|AB| = |Ax|$$

$$\sqrt{9 + (3-x)^2} = 5 \quad /^2$$

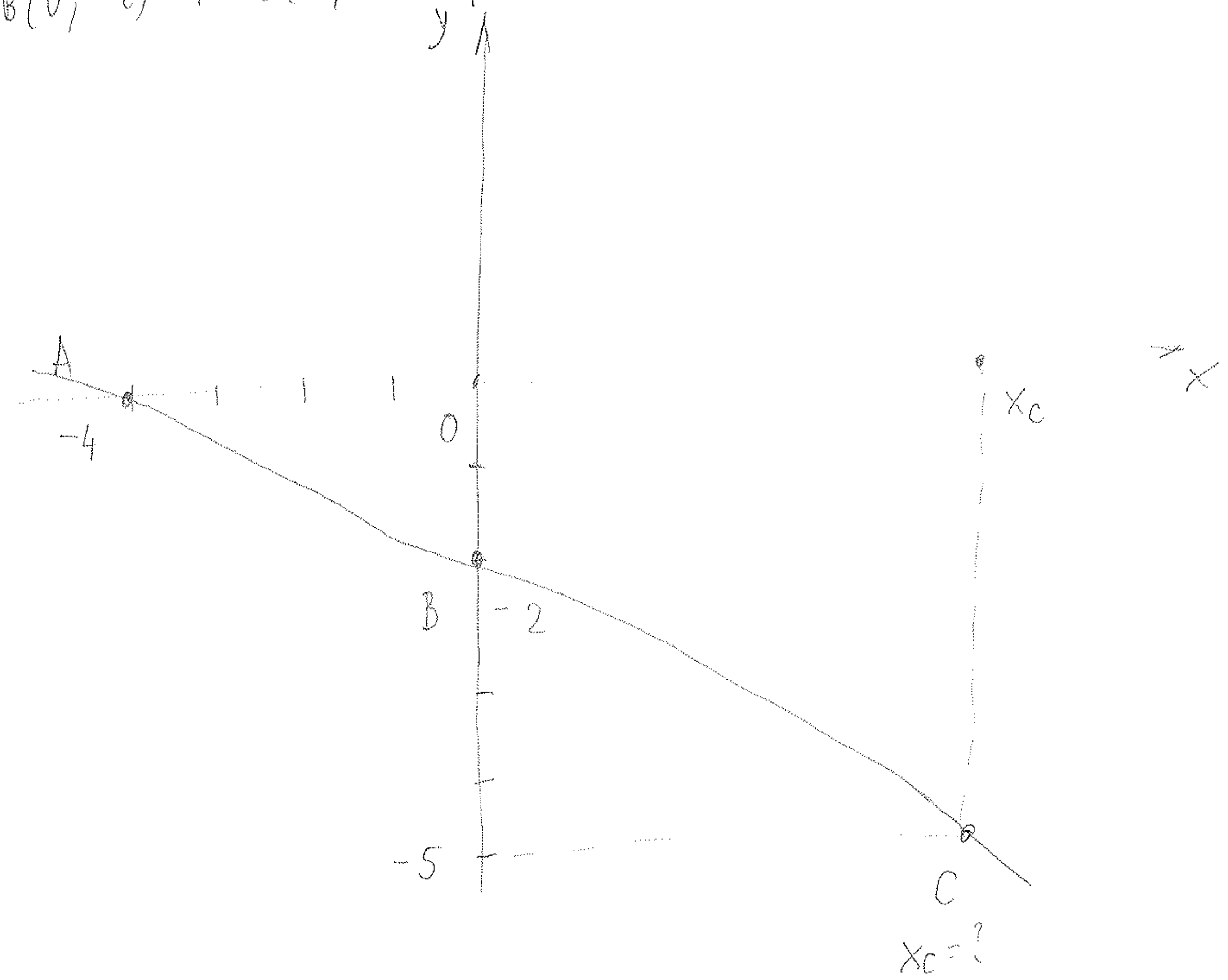
$$9 + (3-x)^2 = 25$$

$$(3-x)^2 = 16 \quad /\sqrt{\ }$$

$$3 - x = 4 \qquad -(3-x) = 4$$

$$\boxed{x = -1} \quad \text{ili} \quad \boxed{x = 7}$$

(57) (zadaci - 13)
Odredite nepoznatu koordinatu točke C ako točke A(-4,0), B(0, -2) i C(x, -5) pripadaju jednom pravcu.



$$y - y_A = \frac{y_B - y_A}{x_B - x_A}(x - x_A)$$

$$y - 0 = \frac{-2 - 0}{0 - (-4)}(x - (-4))$$

$$y = \frac{-2}{4}(x + 4) = -\frac{1}{2}x - 2$$

za točku C:

$$-5 = -\frac{1}{2}x - 2$$

$$-\frac{1}{2}x = -3 \quad / \cdot (-2)$$

$$\boxed{x = 6}$$

DZ

zadaci – 3, 8, 15, 18
ispit 1 – 2, 6, 9
ispit 2 – 1, 9, 10