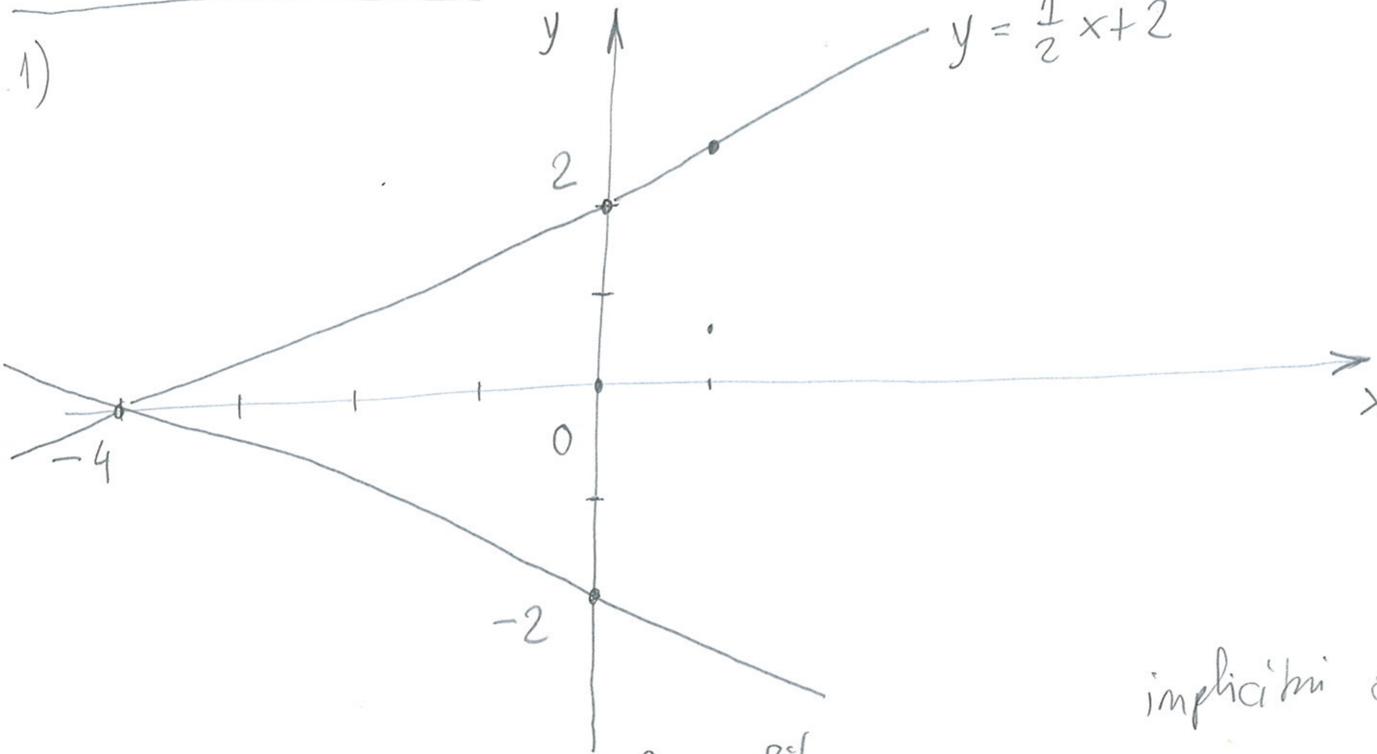
17. PRAVAC

① (Primjer 1)

Napišimo jednadžbu pravca koji je pravcu $x - 2y + 4 = 0$ simetričan:

1) s obzirom na os x,
2) s obzirom na os y,
3) s obzirom na ishodište koordinatnog sustava.

1)



zrcalna slika

$x - 2y + 4 = 0 \implies$ $\boxed{x + 2y + 4 = 0}$ implicitni oblik

$-2y = -x - 4$

$y = \frac{1}{2}x + 2$

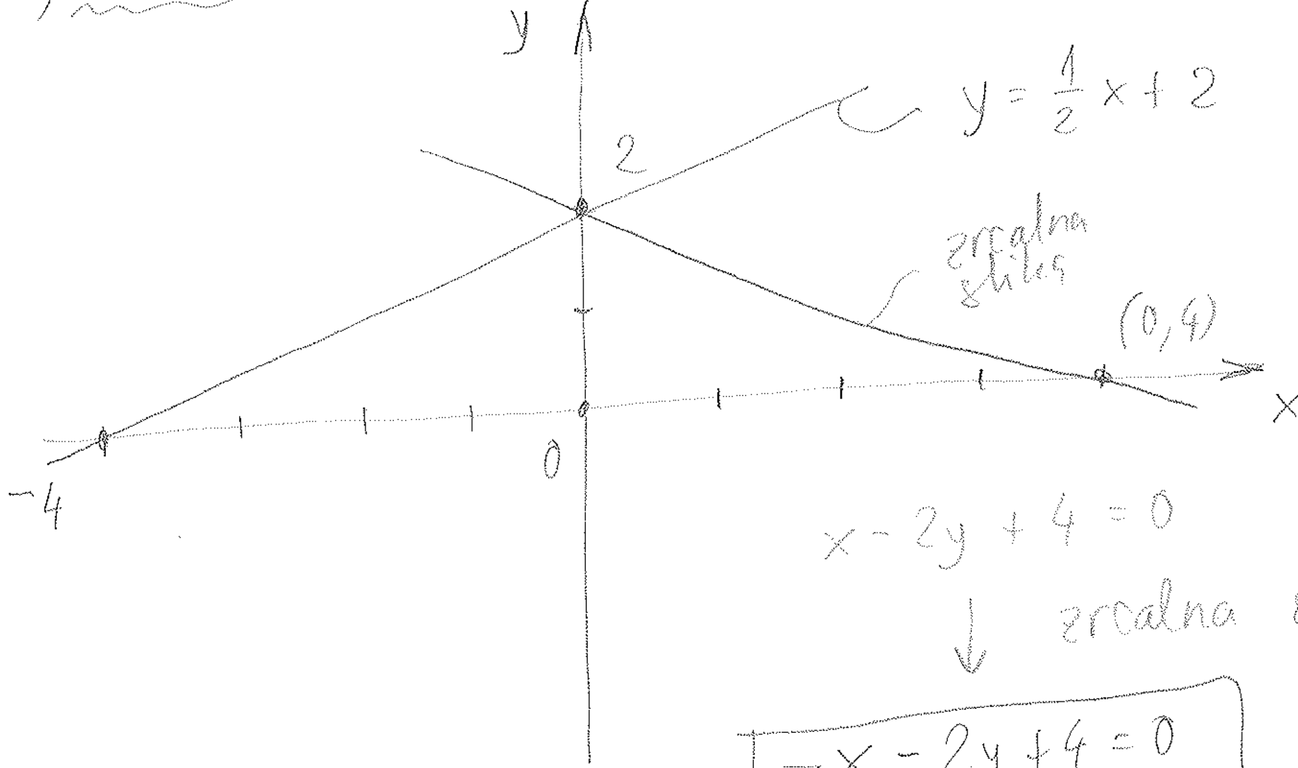
$2y = -x - 4$

$\boxed{y = -\frac{1}{2}x - 2}$ eksplicitni oblik

$\boxed{\begin{array}{c} b \Rightarrow -b \\ a \to -a \end{array}}$

zrcaljenje oko x osi

2) obzirom na os y:



$x - 2y + 4 = 0$

$\downarrow$ zrcalna slika

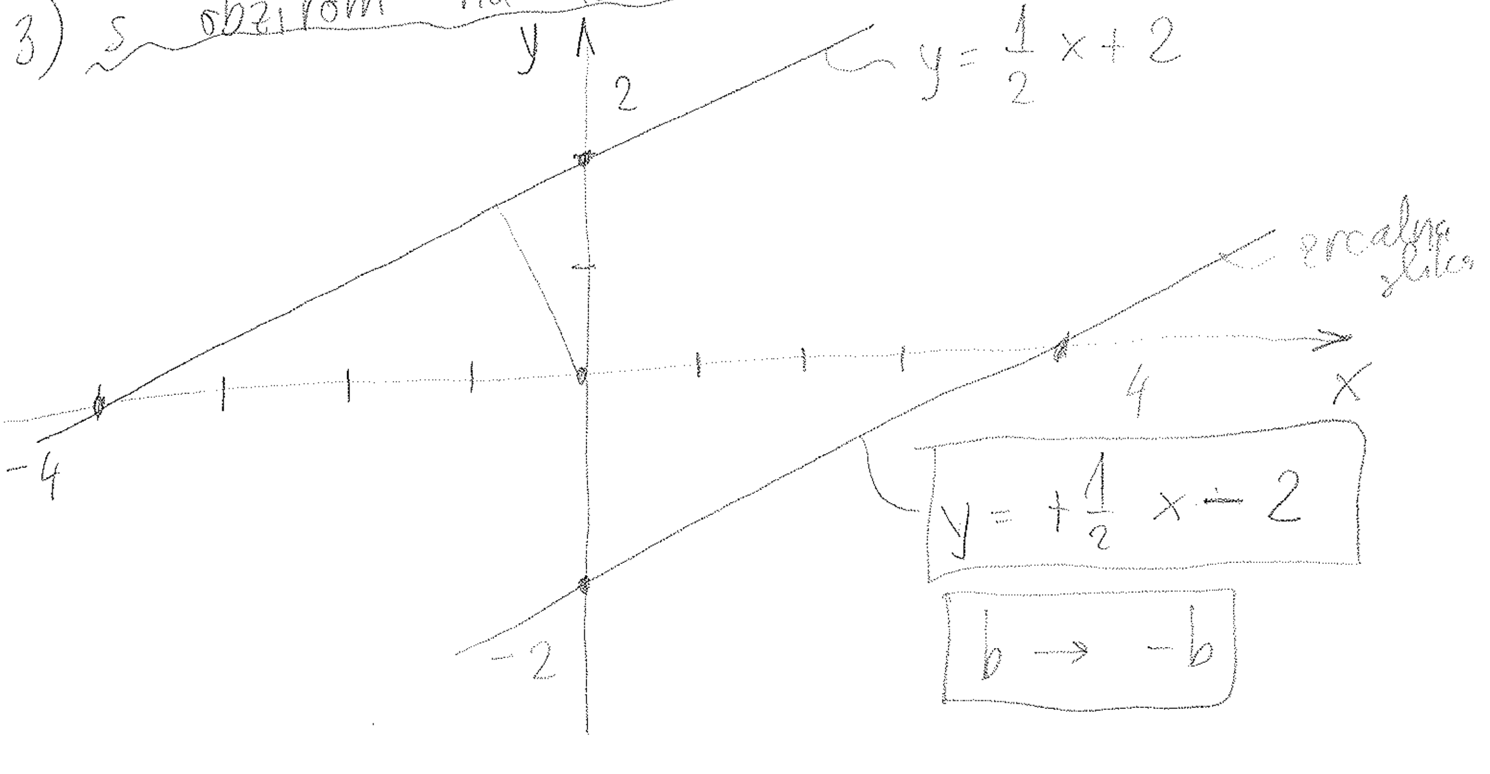
$$\boxed{-x - 2y + 4 = 0}$$

$2y = -x + 4$

$$\boxed{y = -\frac{1}{2}x + 2}$$

$\boxed{a \to -a}$ zrcaljenje oko y osi

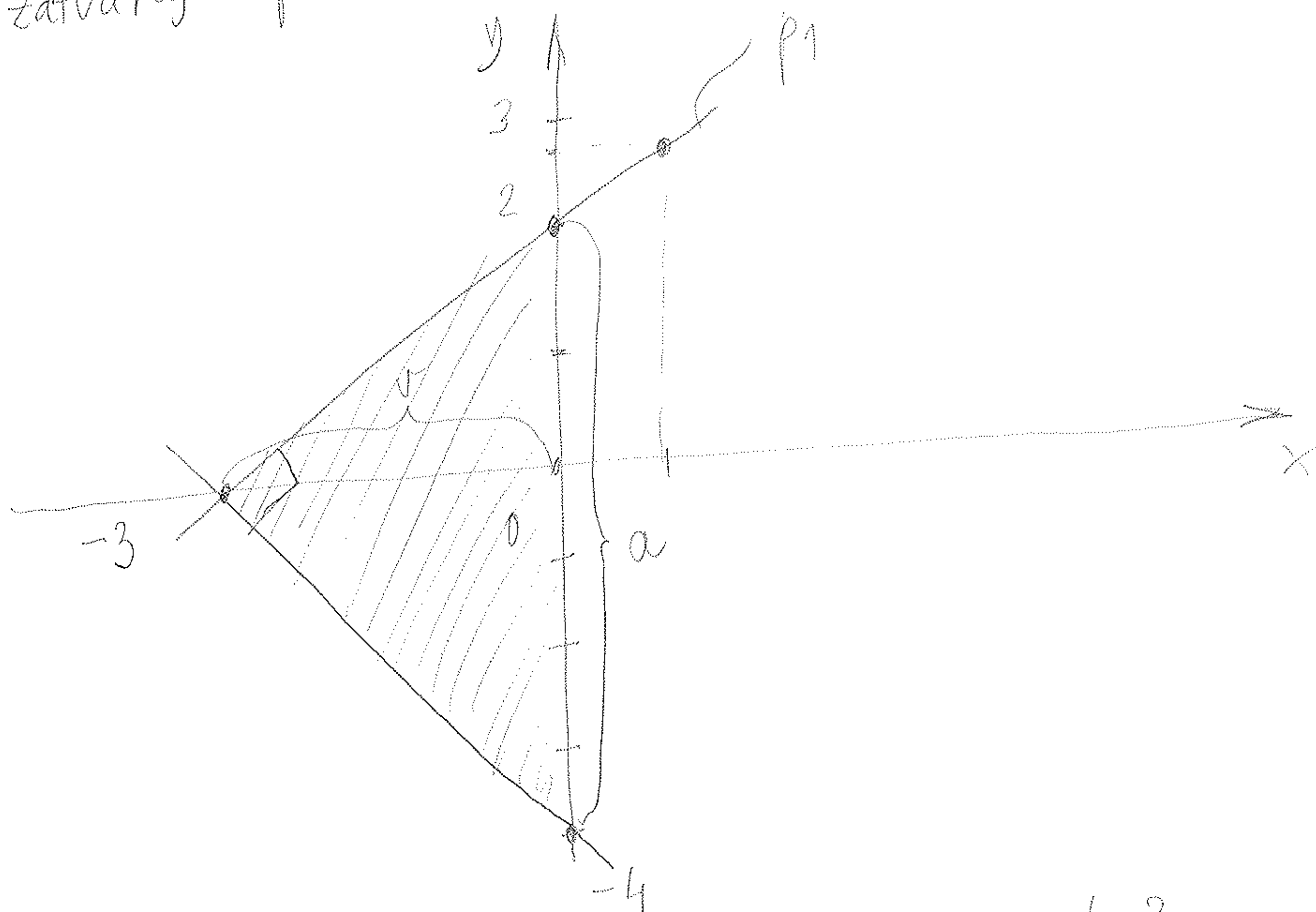
3) s obzirom na ishodište koordinatnog sustava



$$\boxed{y = +\frac{1}{2}x - 2}$$

$$\boxed{b \to -b}$$

② (Primjer 4)
Kolika je površina trokuta što ga s osi ordinata zatvaraju pravci $2x - 3y + 6 = 0$ i $4x + 3y + 12 = 0$?



$p_1 \cdots 2x - 3y + 6 = 0 \Rightarrow 3y = 2x + 6 \;/ : 3$

$y = \frac{2}{3}x + 2$

$p_2 \cdots 4x + 3y + 12 = 0 \Rightarrow 3y = -4x - 12$

$y = -\frac{4}{3}x - 4$

površina trokuta: $\boxed{P = \frac{1}{2} a \cdot v = \frac{1}{2} \cdot 6 \cdot 3 = 9 \text{ kv. jed.}}$

③ (Primjer 3)

Odredimo jednadžbu pravca koji prolazi točkom $T(2,1)$, a s pozitivnim smjerom osi x zatvara dvostruko veći kut od pravca $2x-3y+5=0$.

$p_1 \ldots T(2,1) \ldots y=f(x)=?$ $\qquad \varphi_1 = 2\varphi_2$

$p_2 \ldots 2x-3y+5=0$

$$3y = 2x+5 \;/ :3$$

$$y = \frac{2}{3}x + \frac{5}{3}$$

$$a_2 = k_2 = tg\,\varphi_2$$

$$tg\,\varphi_2 = \frac{2}{3} \Rightarrow \varphi_2 = 33,69^\circ$$

$$\varphi_1 = 2\varphi_2 = 2\cdot 33,69^\circ = 67,38^\circ$$

$$tg\,\varphi_1 = 2,4 = k_1 = a_1$$

$$p_1 \ldots \; y = a_1 x + b_1$$

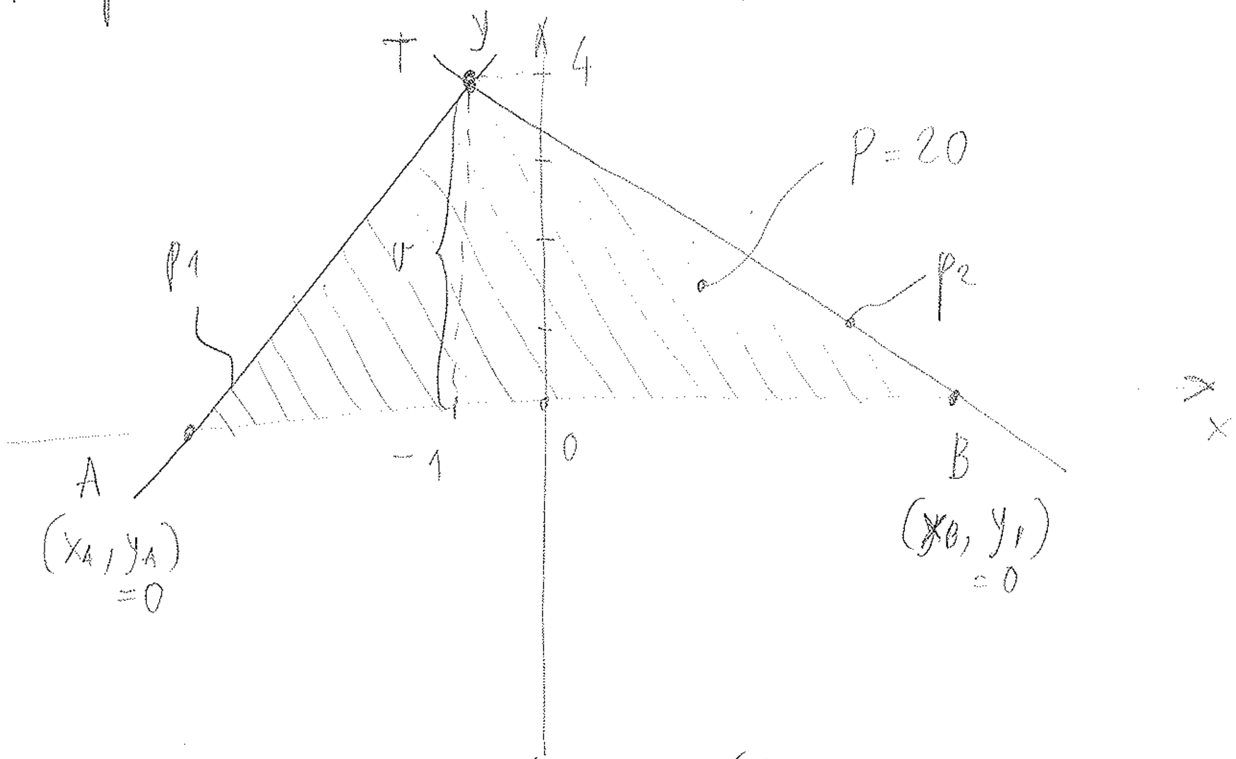
$$y = 2,4x + b_1$$

$$y(x=2) = 1 = 2,4\cdot 2 + b_1$$

$$b_1 = 1 - 4,8 = -3,8$$

$$\boxed{y = 2,4x - 3,8}$$

④ (Primjer 7)
Točkom $T(-1, 4)$ položimo dva okomita pravca koji s osi apcisa zatvaraju trokut površine 20.



pravokutni trokut: $P = \frac{1}{2} a \cdot v$ (1)

$a = |x_B - x_A|$

$v = 4 = y_T \rightarrow (1)$

pretpostavka: $k_1 \ldots p_1$, $k_2 \ldots p_2$ pravci su okomiti pa vrijedi: $k_2 = -\frac{1}{k_1}$

$y = ax + b$

za $x_T = -1$, $y_T = 4$: $\Rightarrow$ $4 = k_1 \cdot (-1) + b_1$ (p_1)

$4 = -k_1 + b_1 \Rightarrow b_1 = 4 + k_1$

za (p_2) $\Rightarrow$ $4 = -\frac{1}{k_1} \cdot (-1) + b_2 \Rightarrow b_2 = 4 - \frac{1}{k_1}$

(p_1) $\ldots$ $y = k_1 x + k_1 + 4$

(p_2) $\ldots$ $y = -\frac{1}{k_1} x - \frac{1}{k_1} + 4$

Odsječci pravaca na osi x:

p_1 ... za $y=0 \Rightarrow 0 = k_1 x + k_1 + 4 \Rightarrow \underset{\sim\sim\sim}{x_1 = -\frac{k_1+4}{k_1}} = x_A$

p_2 ... za $y=0 \Rightarrow 0 = -\frac{1}{k_1}x - \frac{1}{k_1} + 4$

$$\frac{1}{k_1}x = -\frac{1}{k_1} + 4 \quad / \cdot k_1$$

$$\underline{x_2 = -1 + 4k_1} = x_B$$

$$\underline{|x_B - x_A|} = \left| -1 + 4k_1 - \left(-\frac{(k_1+4)}{k_1}\right)\right|$$

$$= \left|\cancel{-1} + 4k_1 + \cancel{1} + \frac{4}{k_1}\right| = \underline{\left|\frac{4k_1^2+4}{k_1}\right|} \rightarrow (1)$$

(1)

$P = \frac{1}{2} a v$

$20 = \frac{1}{2} \cdot \left|\frac{4k_1^2+4}{k_1}\right| \cdot 4$

$20 = 2 \cdot \left|\frac{4k_1^2+4}{k_1}\right| \quad /:2$

$\left|\frac{4k_1^2+4}{k_1}\right| = 10$

$\underbrace{|4k_1^2+4|}_{\text{vrijedi} > 0} = 10|k_1|$

$4k_1^2 + 4 - 10|k_1| = 0 \quad /:2$

$2k_1^2 - 5|k_1| + 2 = 0$

I) $k_1 > 0$

$2k_1^2 - 5k_1 + 2 = 0$

$k_{1_{1/2}} = \frac{5 \pm \sqrt{25-16}}{4}$

$k_{1_{1/2}} = \frac{5 \pm 3}{4}$ ↗ $k_1 = 2$ ↘ $k_2 = \frac{1}{2}$

p_1 ... $y = 2x + 6$

p_2 ... $y = -\frac{1}{2}x + \frac{7}{2}$

II) $k < 0$

$2k_1^2 + 5k_1 + 2 = 0$

$k_{1/2} = \frac{-5 \pm \sqrt{9}}{4}$ ↗ $k_{1_1} = -2$ ↘ $k_{1_2} = -\frac{1}{2}$

p_1 ... $y = -\frac{1}{2}x + \frac{9}{2}$

p_2 ... $y = -2x + 2$

⑤ (Primjer 9)
Stranice kvadrata leže na pravcima $4x-3y+11=0$ i $4x-3y+1=0$. Kolika je površina kvadrata

$p_1 \ldots 4x-3y+11=0 \Rightarrow 3y=4x+11$
$y=\frac{4}{3}x+\frac{11}{3}$ (1)

$p_2 \ldots 4x-3y+1=0 \Rightarrow 3y=4x+1$
$y=\frac{4}{3}x+\frac{1}{3}$ (2)

odabrati točku, npr. $T(1, y_T) \rightarrow (1)$

$$y=\frac{4}{3}+\frac{11}{3}=\frac{15}{3}=5$$

odaljenost točke od pravca: (T od pravca p_2)

$$d=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}$$

$$d=\frac{|4\cdot 1-3\cdot 5+1|}{\sqrt{4^2+(-3)^2}}=\frac{|4-15+1|}{5}=\frac{10}{5}=2$$

$$\boxed{P=d^2=2^2=4 \text{ kv. jed.}}$$

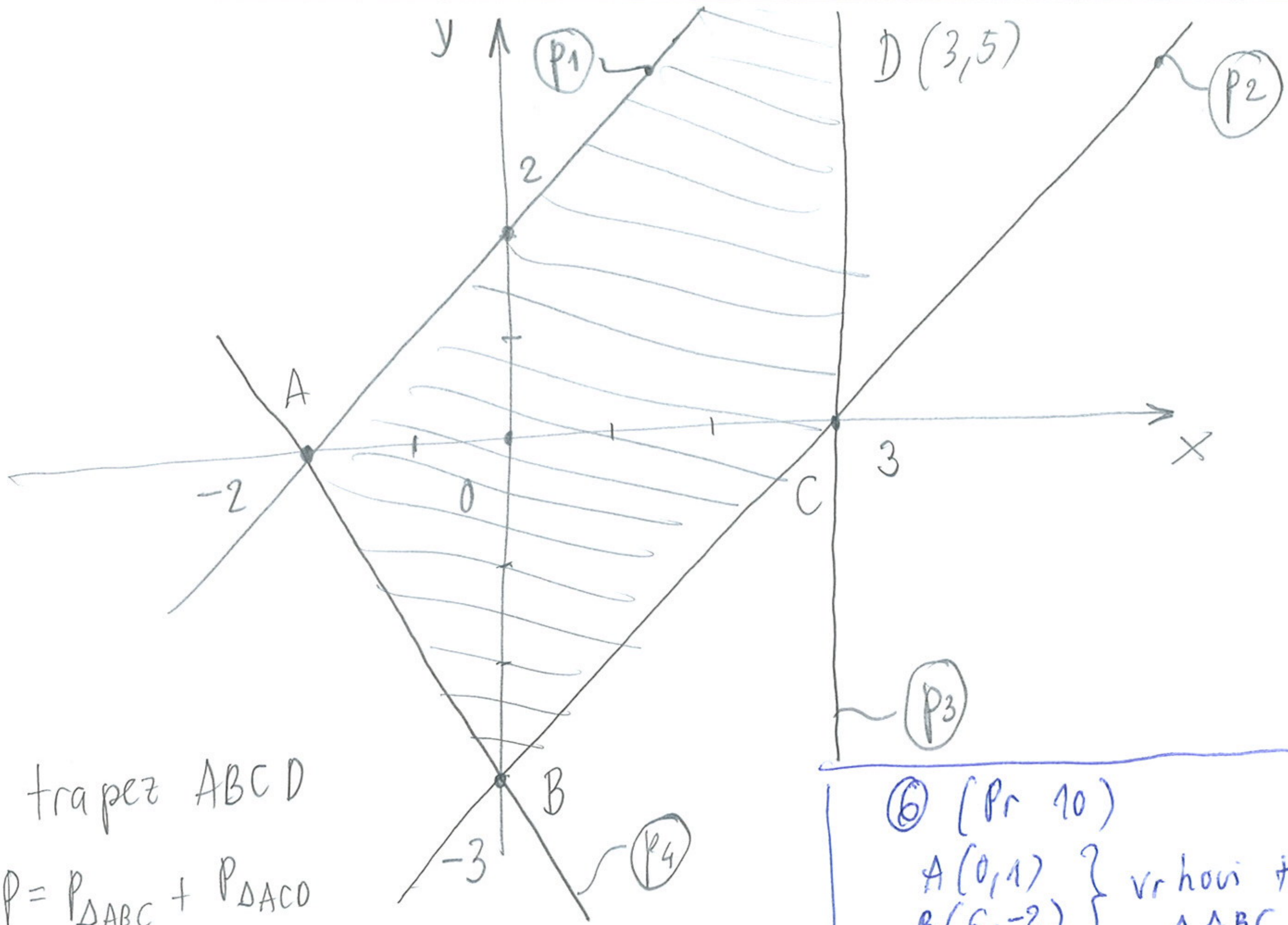
⑦ (Primjer 12)
Kolika je površina dijela ravnine koji je omeđen sustavom nejednadžbi: $3x+2y+6\geq 0$, $x+y+2\geq 0$, $x-y-3\leq 0$, $x-3\leq 0$,

$p_1 \ldots x-y+2\geq 0$
$y\leq x+2$

$p_2 \ldots x-y-3\leq 0$
$y\geq x-3$

$p_3 \ldots x-3\leq 0$
$x\leq 3$

$p_4 \ldots 3x+2y+6\geq 0$
$2y\geq -3x-6 \;/:2$
$y\geq -\frac{3}{2}x-3$



trapez ABCD

$P = P_{\Delta ABC} + P_{\Delta ACD}$

$P_{\Delta ABC} = \frac{5 \cdot 3}{2} = \frac{15}{2}$

$P_{\Delta ACD} = \frac{5 \cdot 5}{2} = \frac{25}{2}$

$\boxed{P = \frac{15}{2} + \frac{25}{2} = \frac{40}{2} = \boxed{20 \text{ kv. jed.}}}$

DZ

zadaci – 5, 6, 9, 13, 16
ispit 1 – 4, 6, 9
ispit 2 – 2, 5, 9, 10

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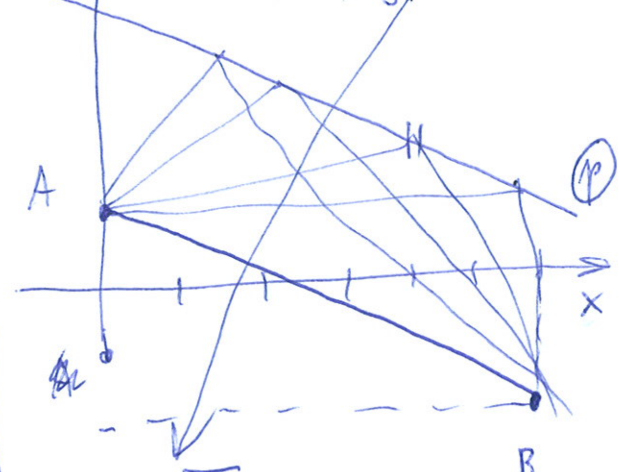
⑥ (Pr 10)

$A(0,1)$ } vrhovi trokuta
$B(6,-2)$ } ΔABC

C ... p ... $mx + 2y - 10 = 0$

površina trokuta se ne mijenja



$p \parallel \overline{AB}$

$k_{AB} = \frac{-2-1}{6-0} = \frac{-3}{6} = -\frac{1}{2}$

$-\frac{1}{2} m = -\frac{1}{2} \Rightarrow m = 1$

$mx + 2y - 10 = 0$

$2y = -mx + 10 \;/ :2$

$y = -\frac{1}{2} m x + 5$

FSB cad lab
Katedra za konstruiranje i razvoj proizvoda

(8) (Pr 11)

$p_1 \ldots 2mx + 3y + 2 = 0 \Rightarrow 3y = -2mx - 2 \,/:3$

$y = -\frac{2}{3}m\,x - \frac{2}{3}$ (k_1)

$p_2 \ldots 2x + my - 1 = 0 \Rightarrow my = -2x + 1 \,/: m$

$y = -\frac{2}{m}x + \frac{1}{m}$ (k_2)

paralelni pravci $k_1 = k_2$

okomiti pravci $k_1 = -\frac{1}{k_2}$

a) paralelni pravci $k_1 = k_2$

$$-\frac{2}{3}m = -\frac{2}{m}$$

$$-\frac{2m}{3} = -\frac{2}{m}$$

$$-2m^2 = -6 \,/:(-2)$$

$$m^2 = 3$$

$$\boxed{m_{1,2} = \pm\sqrt{3}}$$

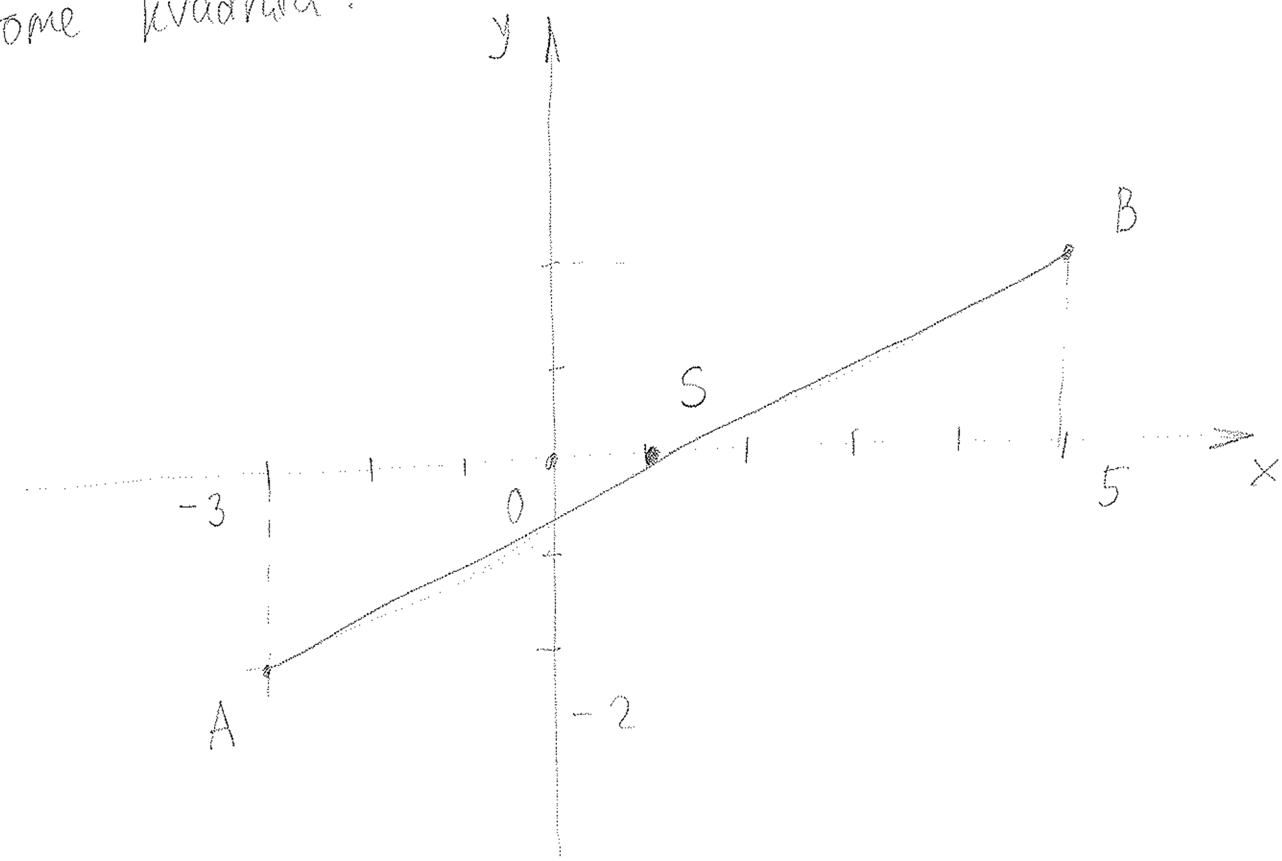
b) okomiti pravci $k_1 = -\frac{1}{k_2}$

$$-\frac{2m}{3} = \frac{m}{2}$$

$-4m = 3m \rightarrow$ pravci ne mogu biti okomiti ni za koju vrijednost m

18. KRUŽNICA

① (Primjer 2)
Dužina $\overline{AC}$, $A(-3,-2)$, $C(5,2)$ dijagonala je kvadrata. Kako glasi jednadžba kružnice opisane, a kako kružnice opisane tome kvadratu?



~~središte kružnice~~ (središte kvadrata):

$S(x_S, y_S)$

$$x_S = \frac{x_A + x_B}{2} = \frac{-3+5}{2} = 1$$

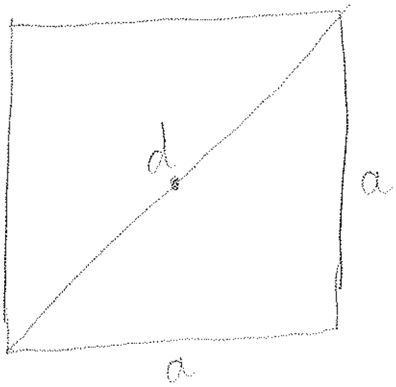
$$y_S = \frac{y_A + y_B}{2} = \frac{-2+2}{2} = 0$$

duljina dijagonale kvadrata:

$$d = \sqrt{(x_C - x_A)^2 + (y_C - y_A)^2} = \sqrt{(5-(-3))^2 + (2-(-2))^2}$$

$$d = \sqrt{64+16} = \sqrt{80} = 4\sqrt{5}$$

duljina stranice kvadrata:



$$d = \sqrt{a^2 + a^2}$$

$$d = \sqrt{2}\,a$$

$$a = \frac{\sqrt{2}}{2} d = \frac{\sqrt{2}}{\cancel{2}_{1}} \cdot \cancel{4}^{2}\sqrt{5} = 2\sqrt{10}$$

polumjer upisane kružnice: $\boxed{\rho = \frac{a}{2} = \frac{2\sqrt{10}}{2} = \sqrt{10}}$

polumjer opisane kružnice: $\boxed{r = \frac{d}{2} = 2\sqrt{5}}$

jednadžba upisane kružnice tom kvadratu:

$$(x - x_S)^2 + (y - y_S)^2 = \rho^2$$

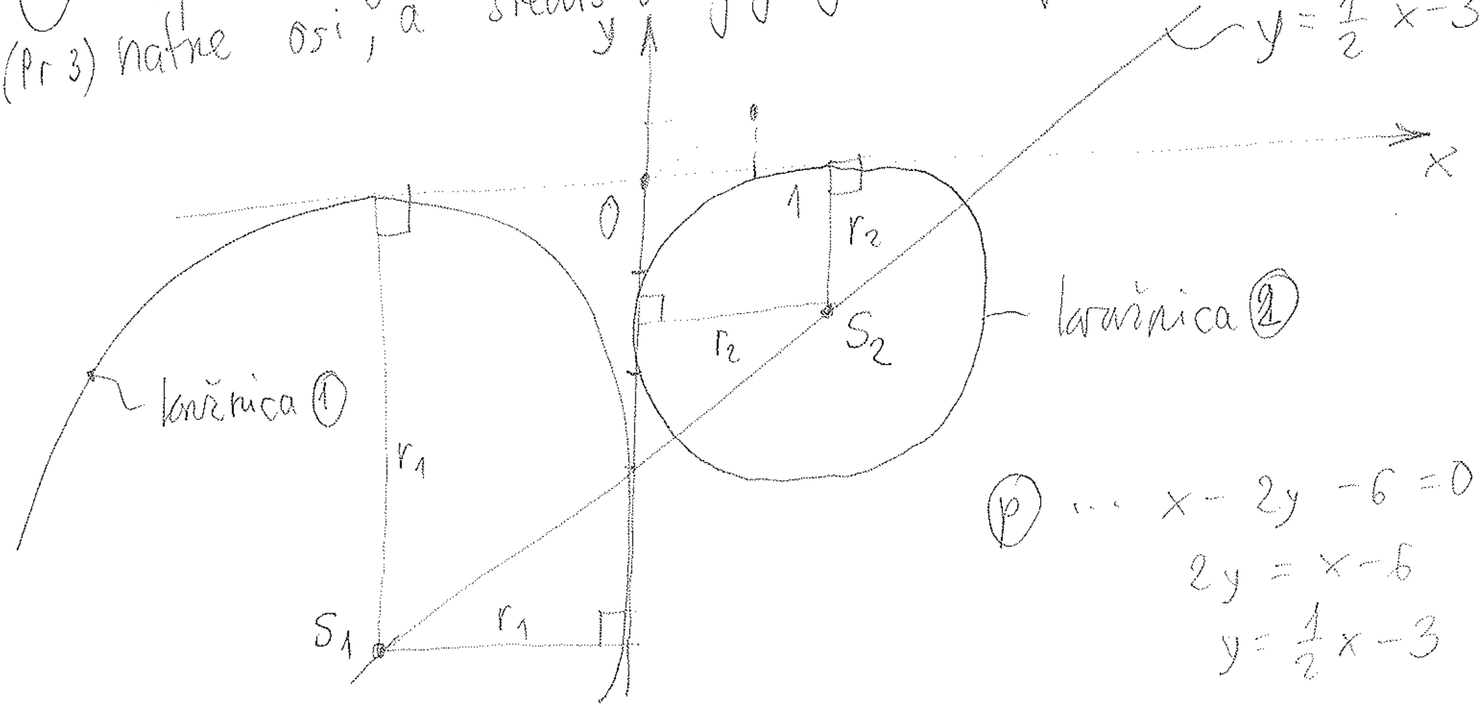
$$\boxed{(x-1)^2 + y^2 = 10}$$

jednadžba opisane kružnice tom kvadratu:

$$(x - x_S)^2 + (y - y_S)^2 = r^2$$

$$\boxed{(x-1)^2 + y^2 = 20}$$

② Napišimo jednadžbu kružnice koja dira obje koordinatne osi, a središte joj je na pravcu $x - 2y - 6 = 0$.
(Pr 3)



p ... $x - 2y - 6 = 0$

$$2y = x - 6$$

$$y = \frac{1}{2}x - 3$$

kružnica .... $(x-p)^2+(y-q)^2=r^2$

$r=|p|=|q|$ – jer kružnica dira obje koordinatne osi

$p-2q-6=0$

$p=2q+6$

$|p|=|q|$

$|p|=2|q|+6=|2q+6|$

rješenja:

I) $p=2q+6$

$q=2q+6$

$q=-6$

$p=-6$

$$\boxed{(x+6)^2+(y+6)^2=36}$$

II) $-p=2q+6$

$-q=2q+6$

$q=-2$

$p=2$

$$\boxed{(x-2)^2+(y+2)^2=4}$$

2' (Pr 5)

$x^2+y^2+2x-4y\overset{+1}{\vee}=0$ ... k_1

k_2 koncentrična kružnici k_1

$(x+2x+1)+(y^2-4y+4)=$ k_2 ... p – tangenta kružnice

$=1+4-1$

$(x+1)^2+(y-2)^2=r_2^2$

$(x+1)^2+(y-2)^2=4$

$S(-1,2)$

$$d(S,p)=\frac{|1\cdot(-1)+(-3)\cdot 2-3|}{\sqrt{1^2+(-3)^2}}$$

$$=\frac{|-1-6-3|}{\sqrt{10}}=\frac{10}{\sqrt{10}}$$

$=\sqrt{10}$

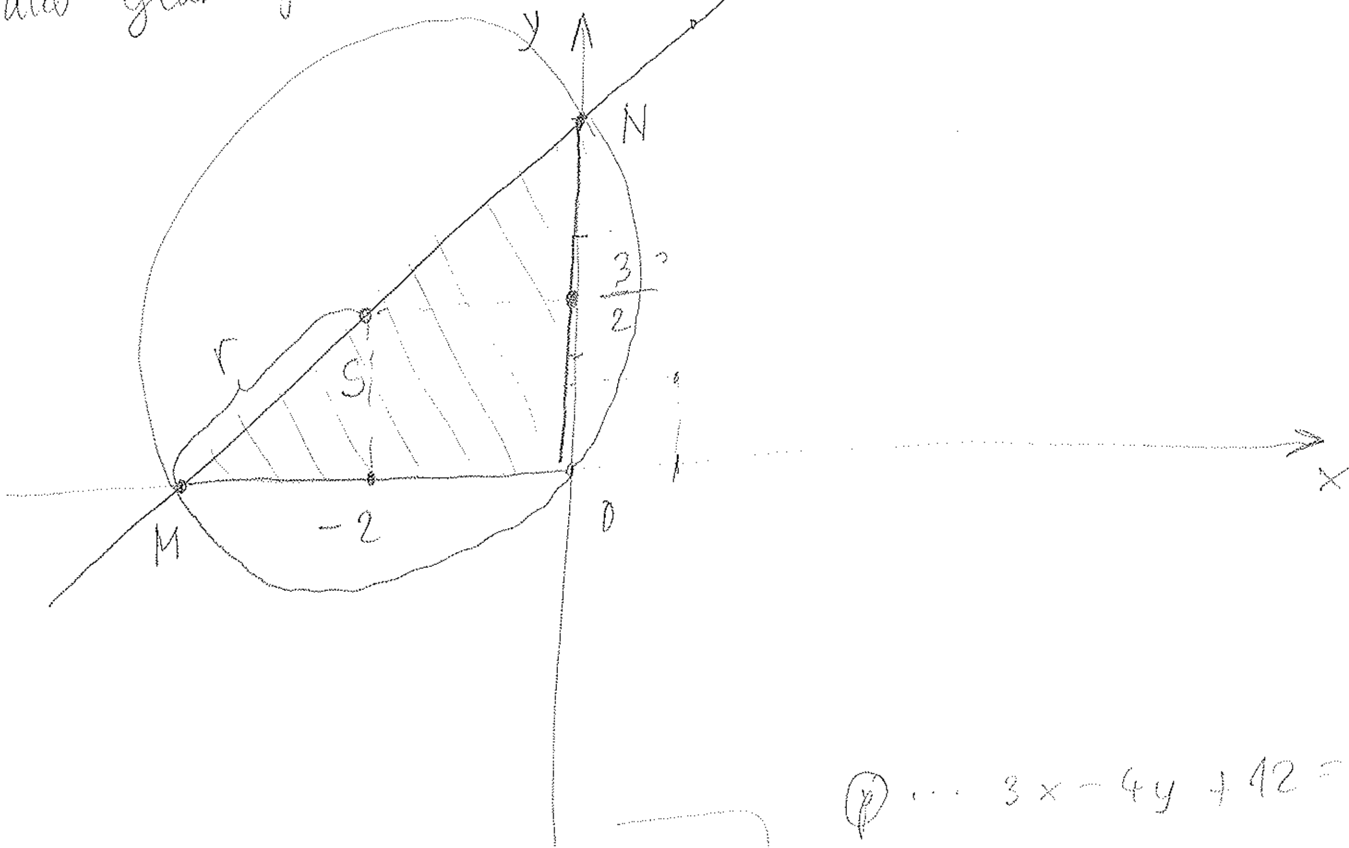
p ... $x-3y-3=0$

$A=1$
$B=-3$
$C=-3$

$\boxed{(x+1)^2+(y-2)^2=10}\rightarrow k_2$

(3) (Primjer 6)

Odredimo jednadžbu kružnice opisane trokutu što ga s koordinatnim osima zatvara pravac $3x-4y+12=0$.
Kako glasi jednadžba kružnice opisane tom trokutu?



$p \ldots \; 3x-4y+12=0$

$4y = 3x+12$

$y = \frac{3}{4}x + 3$

(M) $y=0$

$0 = \frac{3}{4}x + 3$

$-\frac{3}{4}x = 3$

$x = -4$

$M(-4,0)$

(N) $x=0$, $y=3$ $\qquad N(0,3)$

središte trokutu opisane kružnice: → prema Talesovom poučku

S

$$x_S = \frac{x_M + x_N}{2} = \frac{-4+0}{2} = -2$$

$$y_S = \frac{y_M + y_N}{2} = \frac{0+3}{2} = \frac{3}{2}$$

$$S\left(-2, \frac{3}{2}\right)$$

radijus opisane kružnice:

$$r = |MS| = \sqrt{(x_S - x_M)^2 + (y_S - y_M)^2}$$

$$= \sqrt{(-2-(-4))^2 + \left(\frac{3}{2} - 0\right)^2} = \sqrt{4 + \frac{9}{4}}$$

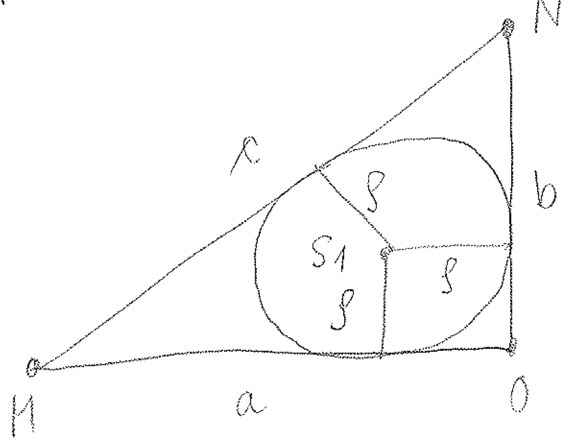
$$= \sqrt{\frac{25}{4}} = \frac{5}{2}$$

jednadžba opisane kružnice:

$$(x - x_S)^2 + (y - y_S)^2 = r^2$$

$$\boxed{(x+2)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{25}{4}}$$

upisana kružnica:



$$P = \rho \cdot s$$

s – poluopseg trokuta

$$s = \frac{1}{2}(a + b + c)$$

$$c = \sqrt{a^2 + b^2}$$

$$c = \sqrt{4^2 + 3^2} = 5$$

$$s = \frac{1}{2}(3 + 4 + 5) = 6$$

$$P = 4 \cdot 3 \cdot \frac{1}{2} = 6$$

$$\rho = \frac{P}{s} = \frac{6}{6} = 1$$

$$\boxed{(x+1)^2 + (y-1)^2 = 1}$$

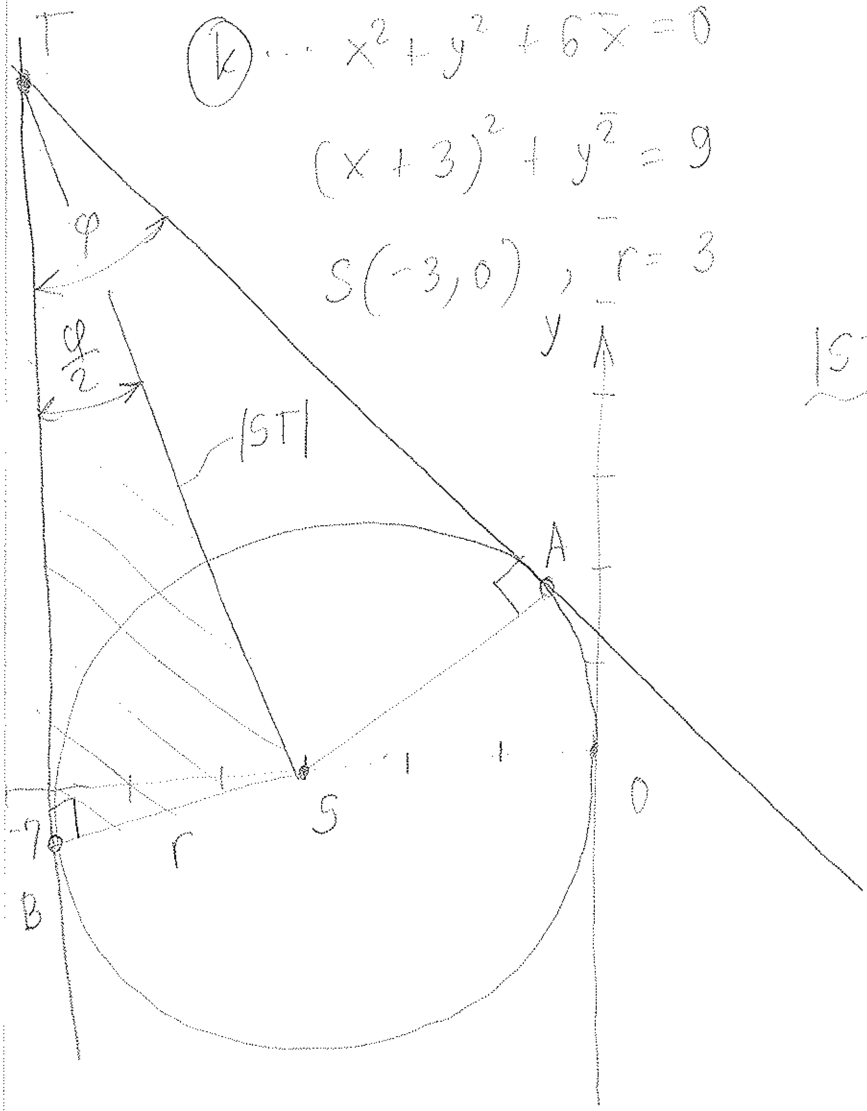
(4) (Primjer 7)

Koliki kut zatvaraju tangente položene iz točke $T(-7,8)$ na kružnicu $x^2+y^2+6x=0$?

(k) ... $x^2+y^2+6x=0$

$(x+3)^2+y^2=9$

$S(-3,0)$, $r=3$



$$|ST| = \sqrt{(-7-(-3))^2+(8-0)^2}$$
$$= \sqrt{(-4)^2+8^2} = 80 = 4\sqrt{5}$$

$$\sin\left(\frac{\varphi}{2}\right) = \frac{r}{|ST|} = \frac{3}{4\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$$
$$= \frac{3\sqrt{5}}{20} = 0,335$$

$$\Rightarrow \frac{\varphi}{2} = 19,597^\circ$$

$$\boxed{\varphi = 39,195^\circ = 39^\circ 11' 42''}$$

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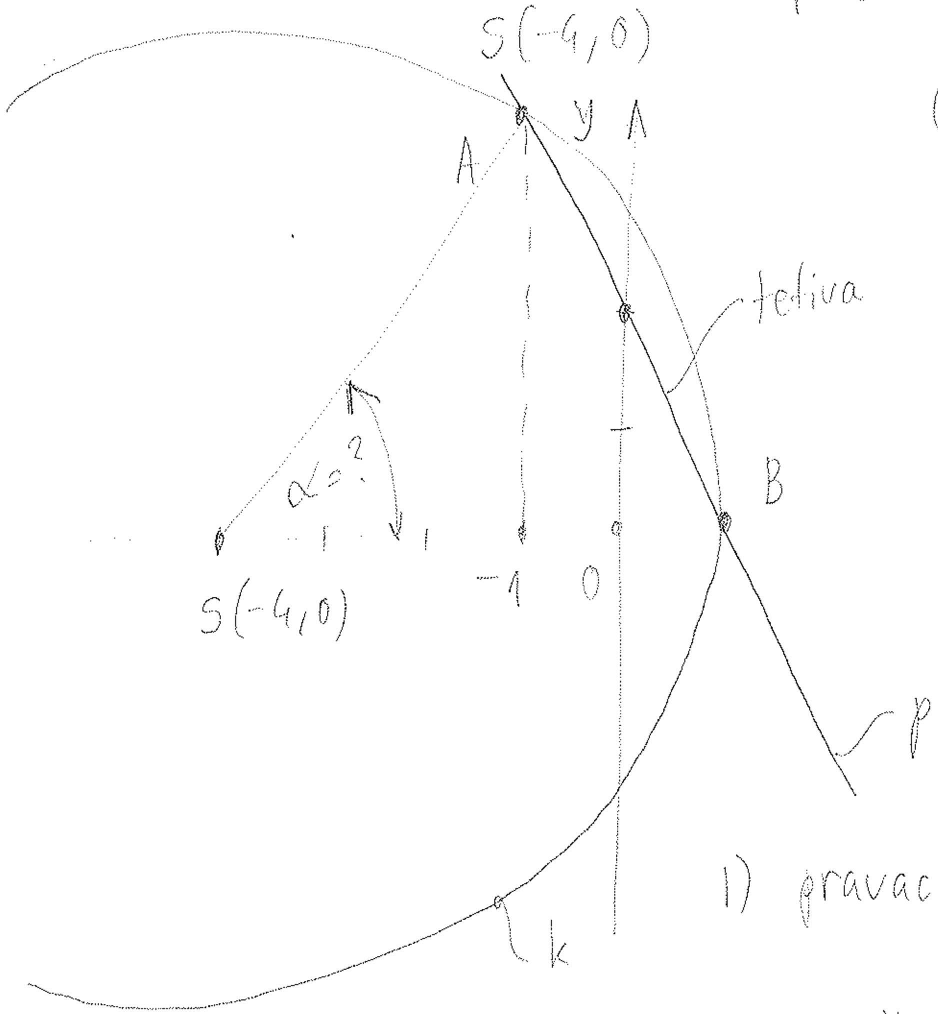
(5) (Primjer 9)
Koliki je središnji kut koji pripada tetivi što je kružnica $x^2+y^2+8x-9=0$ odsijeca na pravcu $2x+y-2=0$?

(k) ... $x^2+y^2+8x-9=0$

$$x^2+8x+16+y^2=9+16$$

$$\underbrace{(x+4)^2+y^2}_{S(-4,0)}=\underbrace{25}_{r=5}$$



(p) ... $2x+y-2=0$

$$y=-2x+2$$

$A(-1,4)$
$B(1,0)$

1) pravac kroz točke S i A:

$$y-y_A=\frac{y_S-y_A}{x_S-x_A}(x-x_A)$$

$$y-4=\frac{0-4}{-4-(-1)}(x-(-1))$$

$$y-4=\frac{-4}{-3}(x+1)$$

$$y=\underbrace{\frac{4}{3}}_{\to \operatorname{tg}\varphi}x+\overbrace{\frac{4}{3}+4}^{\frac{16}{3}}$$

$$\operatorname{tg}\varphi = \frac{4}{3} \Rightarrow \boxed{\varphi = 53{,}13^\circ = 53^\circ 7' 48''}$$

1) $$\sin\varphi = \frac{y_B}{r} = \frac{4}{5} \Rightarrow \varphi = \ldots$$

6 (ispit 1 – 3)

Površina kružnog vijenca jednaka je π. Ako je jednadžba veće kružnice koja ga omeđuje $x^2 + y^2 - 2x + 6y + 1 = 0$, jednadžba manje je:

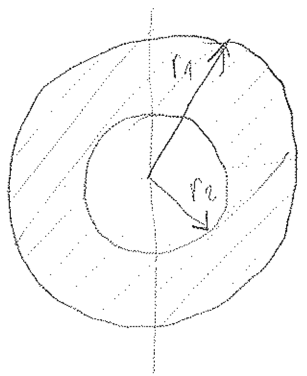
$$k_1 \ldots \quad x^2 + y^2 - 2x + 6y + 1 = 0$$

$$x^2 - 2x + y^2 + 6y = -1$$

$$(x-1)^2 + (y+3)^2 = -1 + 10 = \underbrace{9}_{r}$$

$r = 3$

$S(-1, 3)$



$$P = (r_1^2 - r_2^2)\pi$$

$$\cancel{\pi}^{1} = (r_1^2 - r_2^2)\cancel{\pi}^{1}$$

$$r_2^2 = r_1^2 - 1$$

$$r_2^2 = 9 - 1 = 8$$

$$(x-1)^2 + (y+3)^2 = 8$$

$$x^2 - 2x + 1 + y^2 + 6y + 9 = 8$$

$$k_2 \ldots \quad \boxed{x^2 + y^2 - 2x + 6y + 2 = 0}$$

A

⑰ (ispit 1 – 3)

Pravac $4x - 3y - 1 = 0$ tangenta je kružnice $x^2 + y^2 - 4x + 2y + c = 0$. Tada je c

(t) ... $4x - 3y - 1 = 0$

$$3y = 4x - 1 \quad / :3$$

$$y = \frac{4}{3}x - \frac{1}{3}$$

$y = 0$

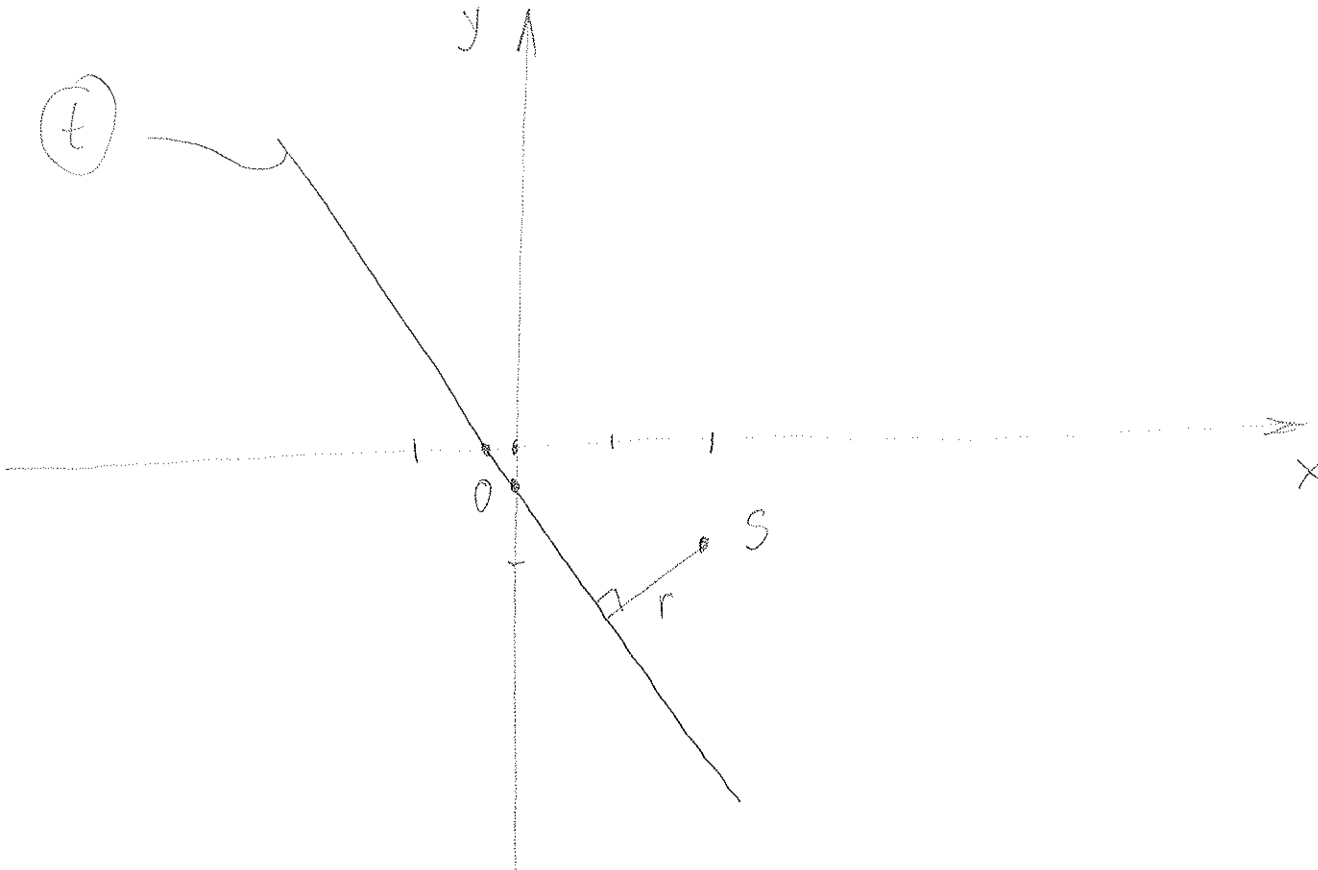
$0 = \frac{4}{3}x - \frac{1}{3}$

$\frac{4}{3}x = \frac{1}{3} \quad / \cdot \frac{3}{4}$

(k) ... $x^2 + y^2 - 4x + 2y + c = 0$

$$(x-2)^2 + (y+1)^2 = -c + 4 + 1 = -c + 5$$

$$S(2, -1)$$



udaljenost točke od pravca:

$$d = r = \frac{|Ax_T + By_T + C|}{\sqrt{A^2 + B^2}} = \frac{|4 \cdot 2 - 3 \cdot (-1) - 1|}{\sqrt{4^2 + (-3)^2}}$$

$$d = \frac{10}{5} = 2$$

$$\left.\begin{aligned} r^2 &= -x+5 \\ r^2 &= 4 \end{aligned}\right\} (=)$$

$$-x+5 = 4$$

$$x = 1$$

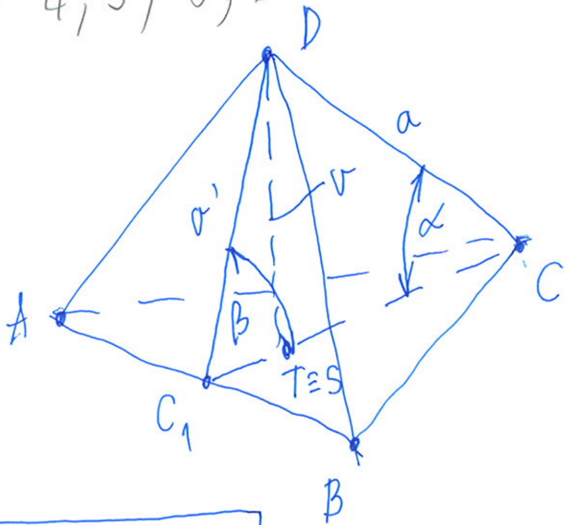
A

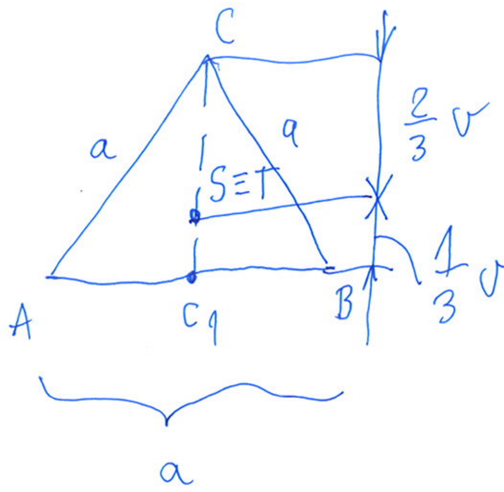
DZ

zadaci - 3, 4, 7, 10, 14, 17

ispit 1 - 4, 6, 7

ispit 2 - 4, 5, 8, 9





$$v' = \sqrt{a^2 - \left(\frac{a}{2}\right)^2}$$

$$v' = \sqrt{a^2 - \frac{a^2}{4}}$$

$$v' = \sqrt{\frac{(4-1)a^2}{4}}$$

$$v' = \frac{\sqrt{3}}{2}a$$

1)

$$v = \sqrt{v'^2 - |C_1S|^2}$$

$$|C_1S| = \frac{1}{3}v' = \frac{1}{3}\cdot\frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{6}a$$

II)

$$v = \sqrt{a^2 - \left(\frac{2}{3}v'\right)^2}$$

$$v = \sqrt{a^2 - \left(\frac{\cancel{2}}{3}\cdot\frac{\sqrt{3}}{\cancel{2}}a\right)^2}$$

$$v = \sqrt{a^2 - \frac{\cancel{3}}{\cancel{9}}a^2} = \sqrt{\frac{2}{3}}a = \frac{\sqrt{6}}{3}a$$

$$v = \sqrt{\left(\frac{\sqrt{3}}{2}a\right)^2 - \left(\frac{\sqrt{3}}{6}a\right)^2}$$

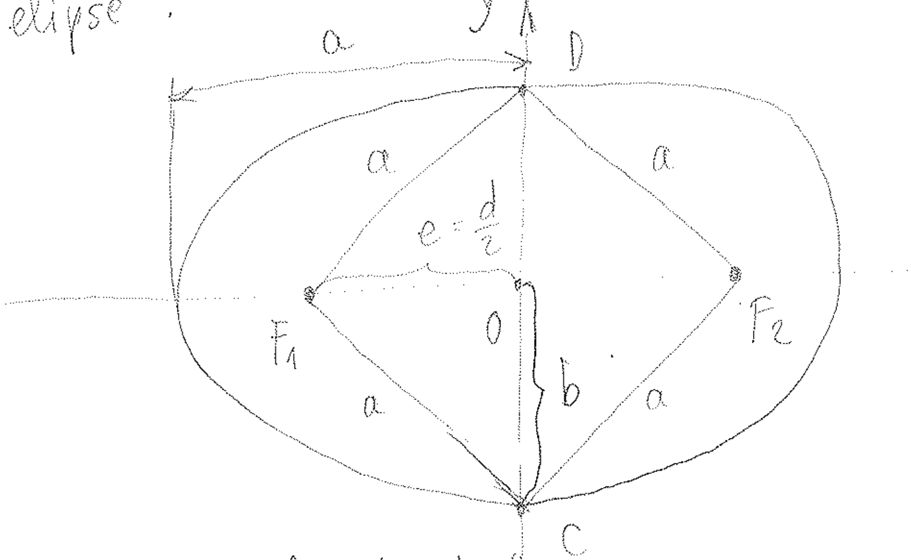
$$= \sqrt{\frac{3}{4}a^2 - \frac{3}{36}a^2} = \sqrt{\frac{9\cdot 3 - 3}{36}}\cdot a = \frac{\sqrt{24}}{6}a = \frac{\sqrt{6}}{3}a$$

$= 2\sqrt{6}$

$\sqrt{24}$

19. ELIPSA, HIPERBOLA I PARABOLA

① (Primjer 3)
Žarišta elipse $F_1(-4,0)$ i $F_2(4,0)$, te krajnje točke C i D male osi elipse vrhovi su kvadrata. Kako glasi jednadžba ove elipse?



$a = a$
↳ stranica upisanog kvadrata
↳ velika poluos

$\overline{F_1F_2}$ – dijagonala kvadrata

$|F_1F_2| = 8 = 2e = d \Rightarrow \underline{e = 4}$

$$a^2 + a^2 = d^2$$

$$2a^2 = d^2$$

$$a^2 = \frac{d^2}{2} = \frac{8^2}{2} = 32$$

$$a = \frac{d}{\sqrt{2}} = \frac{2e}{\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2}$$

$$|CD| = 2b = 8 \Rightarrow b = 4$$

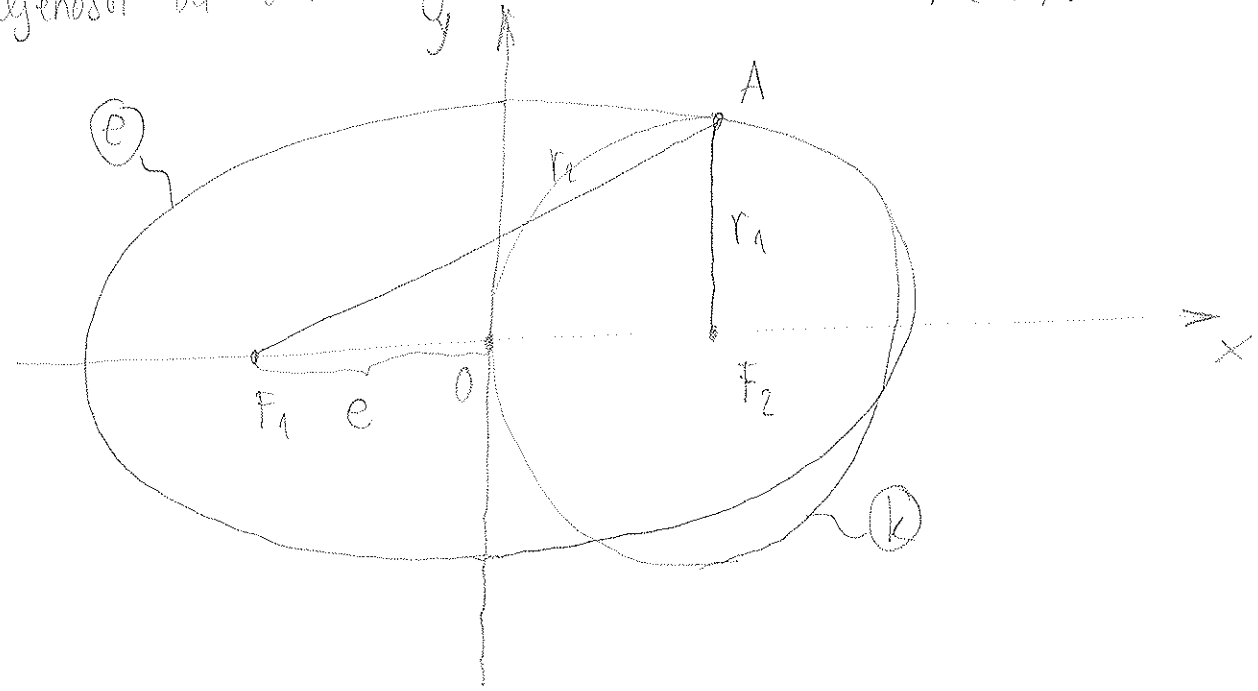
$$b^2 = 16$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\boxed{\frac{x^2}{32} + \frac{y^2}{16} = 1}$$

② (Primjer 4)
Odredimo onu točku elipse $5x^2+9y^2=405$ za koju je razlika udaljenosti od žarišta jednaka 8.

$A(x_A, y_A) = ?$



$$e \ldots \; 5x^2 + 9y^2 = 405 \quad / : 405$$

$$\frac{x^2}{81} + \frac{y^2}{45} = 1 \quad \Rightarrow \quad a^2 = 81 \qquad b^2 = 45 = 9 \cdot 5$$

$$a = 9 \qquad b = 3\sqrt{5}$$

$$r_1 + r_2 = 2a$$

za točku A: $r_1 + r_2 = 18$

zadano: $r_1 - r_2 = 8$ $\Big\}$ $(+)$

$$2r_1 = 26 \quad \Rightarrow \quad r_1 = 13$$

$$r_2 = 5$$

$$a^2 - b^2 = e^2 = 36 \quad \Rightarrow \quad e = 6$$

$$F_1(-6, 0)$$

$$F_2(6, 0)$$

1) $k \ldots \; (x-6)^2 + y^2 = 25$

$$5x^2 + 9y^2 = 405 \quad \Rightarrow \quad 9y^2 = 405 - 5x^2 \quad / : 9$$

$$y^2 = 45 - \frac{5}{9}x^2$$

$$(x-6)^2 + 45 - \frac{5}{9}x^2 = 25$$

$$x^2 - 12x + 36 + 45 - \frac{5}{9}x^2 = 25$$

$$\frac{4}{9}x^2 - 12x + 56 = 0$$

rješenja: $\boxed{\begin{matrix} T_1(6,5) \\ T_2(6,-5) \end{matrix}}$ te zbog simetričnosti elipse oko osi y $\boxed{\begin{matrix} T_3(-6,5) \\ T_4(-6,-5) \end{matrix}}$

~~II)~~ $r_1^2 - r_2^2 = (2e)^2$

⋮

(3) (Primjer 6)

Odredimo skup točaka ravnine zadanih jednadžbom
$x^2 - y^2 - 4x - 6x - 9 = 0$.

(R) .... $x^2 - y^2 - 4x - 6y - 9 = 0$

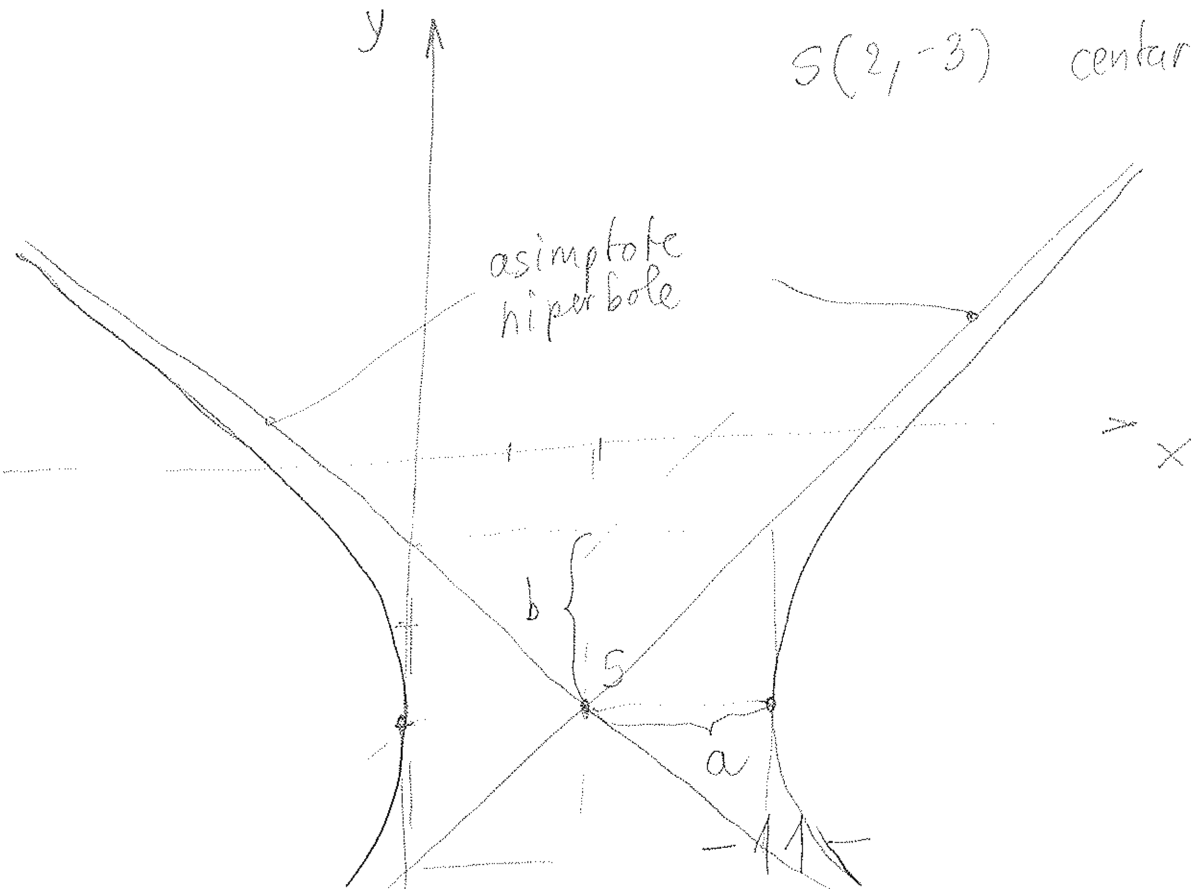
$$(x-2)^2 - (y+3)^2 = 9 + 4 - 9 = 4 \quad /:4$$

$$\frac{(x-2)^2}{4} - \frac{(y+3)^2}{4} = 1$$

↳ hiperbola: $\dfrac{(x-x_S)^2}{a^2} - \dfrac{(y-y_S)^2}{b^2} = 1$

$a = b = 2$ jednakostranična hiperbola

$S(2,-3)$ centar simetrije



④ (Primjer 7)

Odredimo onu točku hiperbole $9x^2 - 16y^2 = 144$ čije su spojnice sa žarištima hiperbole međusobno okomite.

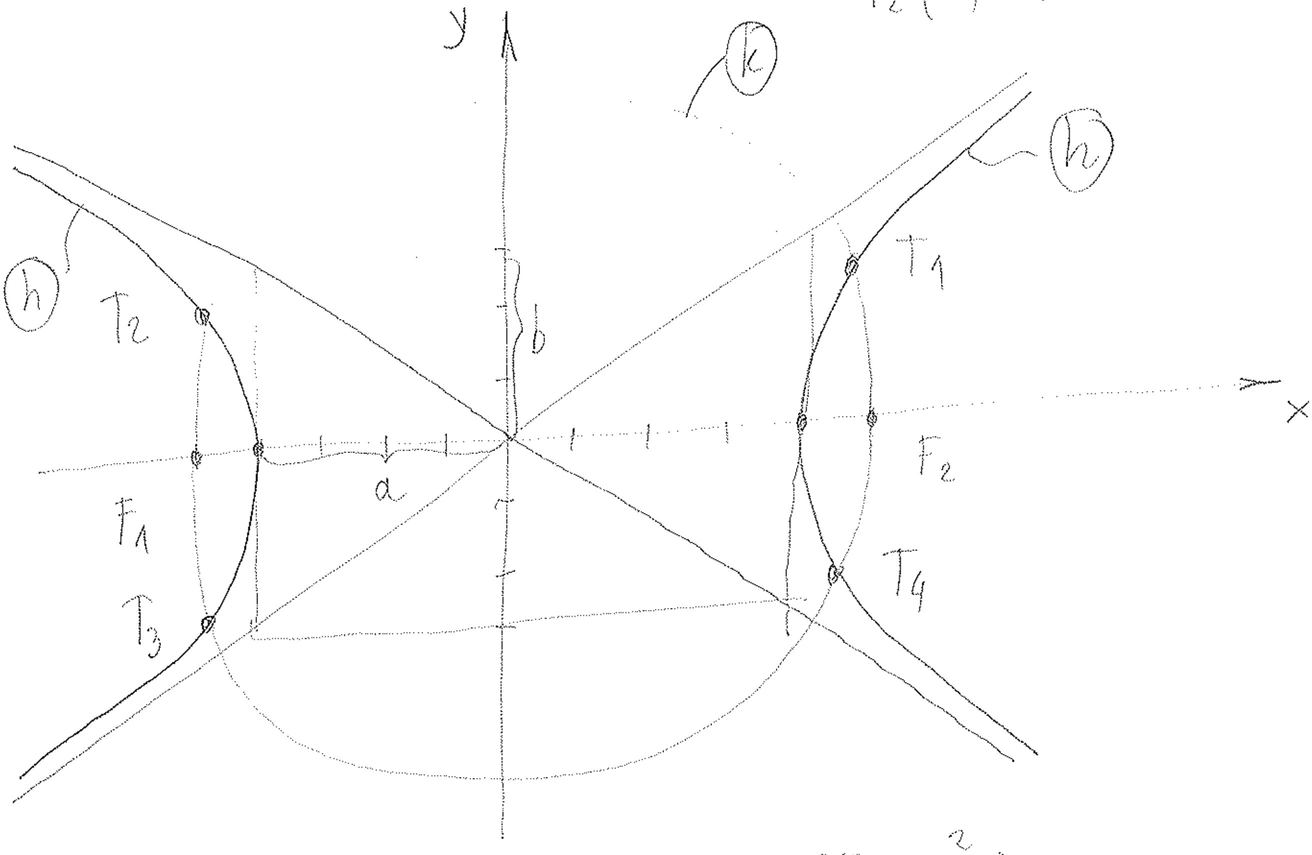
$$(h) \ldots \quad 9x^2 - 16y^2 = 144 \quad / : 144$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \longrightarrow a^2 = 16 \quad b^2 = 9$$

$$a = 4 \quad b = 3$$

žarišta hiperbole: $\quad e^2 = a^2 + b^2 = 16 + 9 = 5$

$$\Rightarrow F_1(-5, 0)$$
$$F_2(5, 0)$$



$$(k) \ldots \quad x^2 + y^2 = 25 \Rightarrow x^2 = 25 - y^2$$

$$9(25 - y^2) - 16y^2 = 144$$

$$225 - 9y^2 - 16y^2 = 144$$

$$-25y^2 = 144 - 225 = -81$$

$$y^2 = \frac{81}{25}$$

$$y_1 = \frac{9}{5}, \quad y_2 = -\frac{9}{5}$$

rješenja:

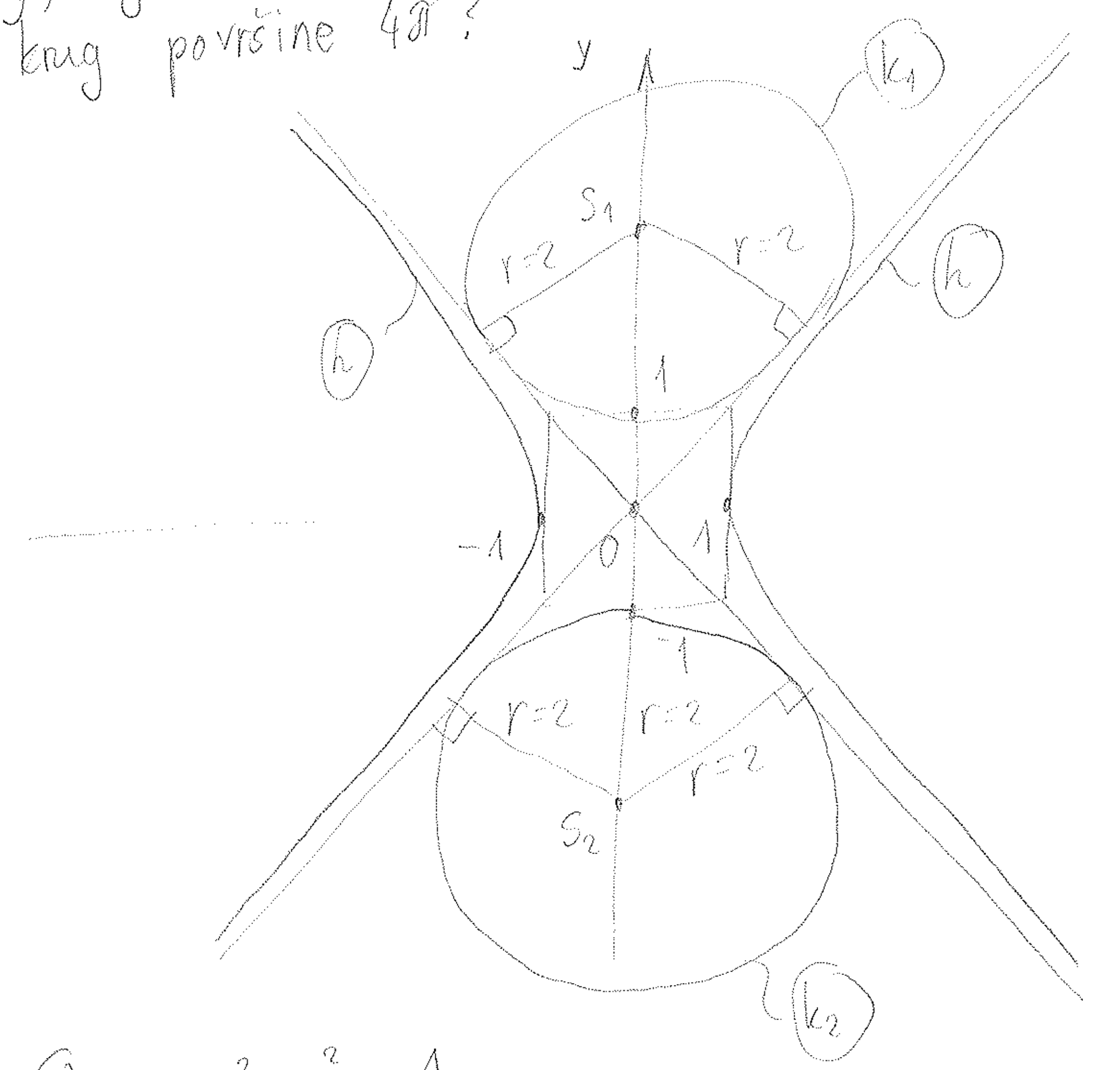
$T_1\left(\frac{4\sqrt{34}}{5}, \frac{9}{5}\right)$

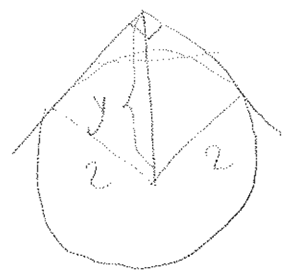
$T_2\left(-\frac{4\sqrt{34}}{5}, \frac{9}{5}\right)$

$T_3\left(-\frac{4\sqrt{34}}{5}, -\frac{9}{5}\right)$

$T_4\left(\frac{4\sqrt{34}}{5}, -\frac{9}{5}\right)$

5 (Primjer 8)
Kako glasi jednadžba kružnice kojoj je središte na osi y, koja dira asimptote hiperbole $x^2 - y^2 = 1$ i koja omeđuje krug površine 4π?





$y^2 = 2^2 + 2^2 = 8$

$\underline{y_1 = 2\sqrt{2}}$

$\underline{y_2 = -2\sqrt{2}}$

h ... $x^2 - y^2 = 1$

$\underline{a = b = 1}$

k ... $P = 4\pi = r^2\pi \Rightarrow r = 2$

$x^2 + (y-2)^2 = 4$

$S(0,2)$ - središte kružnice

dva rješenja:

1) ... $x^2 + (y - 2\sqrt{2})^2 = 4$ k_1

2) ... $x^2 + (y + 2\sqrt{2})^2 = 4$ k_2

⑥ (ispit 1 – 5)

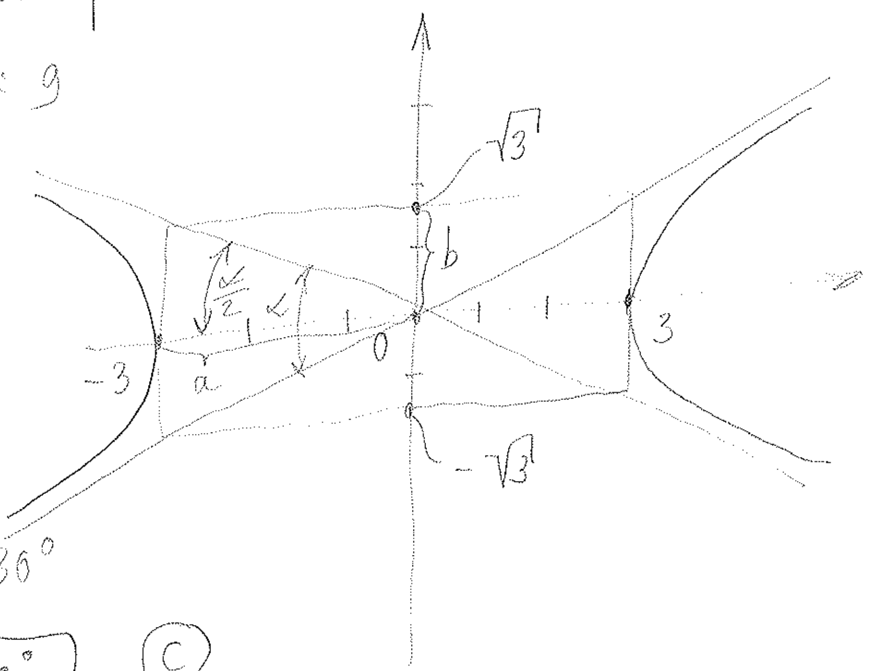
Kut što ga zatvaraju asimptote hiperbole $x^2 - 3y^2 = 9$ iznosi:

(h) ... $x^2 - 3y^2 = 9 \;/ : 9$

$\frac{x^2}{9} - \frac{y^2}{3} = 1$

$\hookrightarrow b = \sqrt{3}$

$\hookrightarrow a = 3$



$tg\left(\frac{\alpha}{2}\right) = \frac{\sqrt{3}}{3} \Rightarrow \frac{\alpha}{2} = 30^\circ$

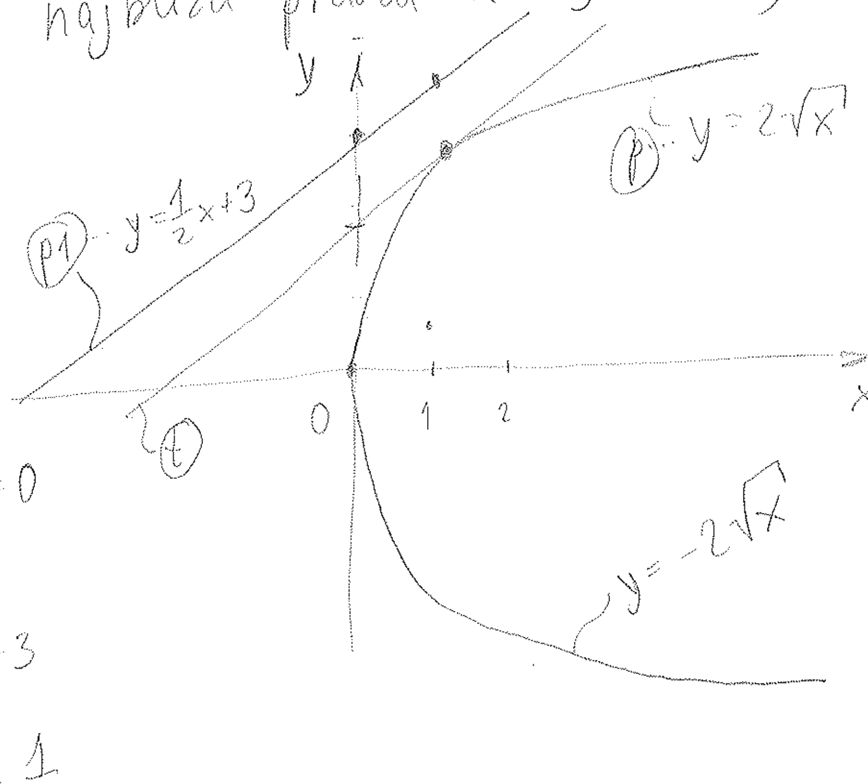
$\boxed{\alpha = 60^\circ}$ (c)

⑦ Točka parabole $y^2 = 4x$ koja je najbliža pravcu $x - 2y + 6 = 0$,

(ispit 1 – 8) jest točka:

(p) ... $y^2 = 4x$

$y = 2\sqrt{x}$



(p1) ... $x - 2y + 6 = 0$

$2y = x + 6$

$y = \frac{1}{2}x + 3$

(t) ... $a_T = a_1 = \frac{1}{2}$

$y = \frac{1}{2}x + b_T$

$y^2 = 4x = 2px \Rightarrow 2p = 4$

$p = 2$

uvjet dodira pravca i parabole: $p = 2a_T b_T$

$2 = 2 \cdot \frac{1}{2} \cdot b_T \Rightarrow b_T = 2$

$y = \frac{1}{2}x + 2$

$$y^2 = 4x$$
$$y = \frac{1}{2}x + 2$$

$$\left(\frac{1}{2}x + 2\right)^2 = 4x$$

$$\frac{1}{4}x^2 + 2x + 4 = 4x$$

$$\frac{1}{4}x^2 - 2x + 4 = 0 \quad / \cdot 4$$

$$x^2 - 8x + 16 = 0$$

$$(x-4)^2 = 0$$

$$x_{1,2} = 4 \Rightarrow y = \frac{1}{2} \cdot 4 + 2 = 4$$

$$\boxed{T(4,4)}$$

(B)

DZ zadaci – 3, 6, 8, 11, 12, 17, 19
ispit 1 – 1, 4, 6
ispit 2 – 4, 5, 8, 10
ispit 3 – 7, 8, 9
ispit 4 – 4, 6, 8, 10

(8) (Pr 11)

Dana je parabola $y = \underbrace{x^2}_{\to a=1} - 2x - 3$. Odredimo žarište i ravnalicu te parabole.

ravnalica – direktrisa parabole – d

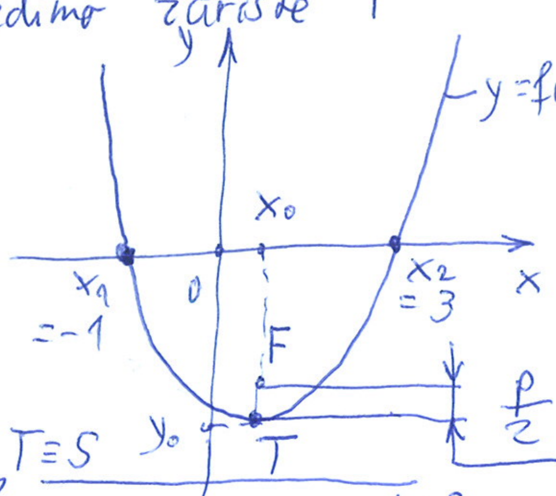
žarište (fokus) parabole – F

↳ $F\left(x_0, -\frac{p}{2} + y_0\right)$

koordinate tjemena parabole $S(x_0, y_0)$

$$a = \frac{1}{2p} \Rightarrow p = \frac{1}{2}$$

↳ parametar parabole



$T \equiv S$

$$x_0 = \frac{x_1 + x_2}{2} = \frac{-1+3}{2} = 1$$

$$y_0 = f(x_0) = 1^2 - 2 \cdot 1 - 3 = -$$

$$\boxed{F\left(1, -\frac{15}{4}\right)} \text{ fokus}$$

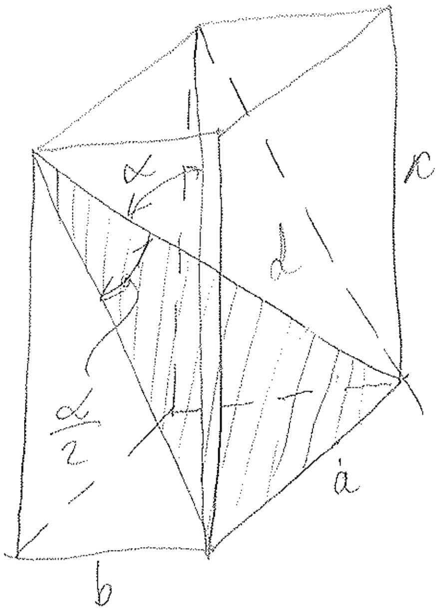
(d) ... $y = -\frac{p}{2} + y_0$

$$y = -\frac{1}{4} - 4 = \frac{-1-16}{4}$$

$$\boxed{y = -\frac{17}{4}} \text{ direktrisa}$$

20. GEOMETRIJA PROSTORA

① (Primjer 2)
Koliki kut zatvaraju prostorne dijagonale u najmanjem dijagonalnom presjeku kvadra ako su duljine bridova kvadra jednake 9 cm, 12 cm i 20 cm?



$a = 9$ cm

$b = 12$ cm

$c = 20$ cm

d – prostorna dijagonala kvadra

$$d = \sqrt{a^2 + b^2 + c^2}$$

$$d = \sqrt{9^2 + 12^2 + 20^2}$$

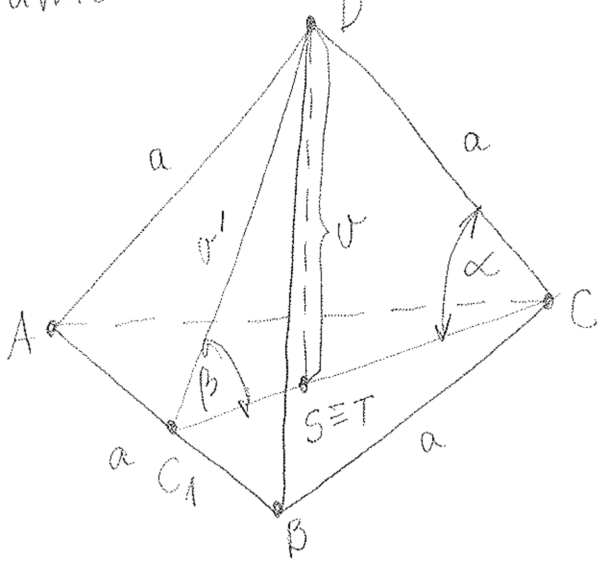
$$= \sqrt{81 + 144 + 400}$$

$$= 25 \text{ cm}$$

$$\sin\left(\frac{\alpha}{2}\right) = \frac{a}{d} = \frac{9}{25} \Rightarrow \frac{\alpha}{2} = \arcsin\left(\frac{9}{25}\right) = 21{,}1^\circ$$

$$\boxed{\alpha = 42{,}2^\circ = 42^\circ 12'}$$

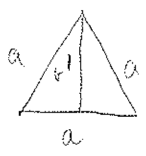
② (Primjer 4)
Svi bridovi pravilne trostrane piramide jednake su duljine. Koliki je priklonı kut bočnog brida prema ravnini osnovke piramide? Koliki kut zatvaraju ravnine dviju strana piramide



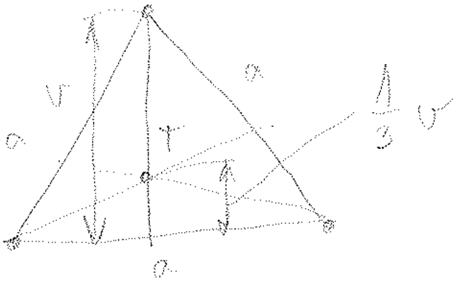
α – priklonı kut bočnog brida prema ravnini osnovke piramide

$$\sin\alpha = \frac{a}{v}$$

$$v = \sqrt{v'^2 - |C_1S|^2}$$



$$v' = \sqrt{a^2 - \frac{a^2}{4}} = \frac{\sqrt{3}}{2}a$$



za jednakostranični trokut

$|G_1S| = \frac{1}{3}v' = \frac{1}{3}\cdot\frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{6}a$

$$v = \sqrt{a^2 - \left(\frac{2}{3}\cdot\frac{\sqrt{3}}{2}a\right)^2} = \sqrt{a^2 - \frac{1}{3}a^2} = \sqrt{\frac{2}{3}}a\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}a$$

$$\sin\alpha = \frac{v}{a} = \frac{\frac{\sqrt{6}}{3}a}{a} = \frac{\sqrt{6}}{3} \Rightarrow \boxed{\alpha = 54{,}736^\circ = 54^\circ 44'}$$

$$\sin\beta = \frac{v}{v'} = \frac{\frac{\sqrt{6}}{3}a}{\frac{\sqrt{3}}{2}a} = \frac{\sqrt{6}}{3}\cancel{a}\cdot\frac{2}{\sqrt{3}\cdot\cancel{a}} = \frac{\sqrt{2}\cdot\cancel{\sqrt{3}}\cdot 2}{3\cancel{\sqrt{3}}} = \frac{2\sqrt{2}}{3}$$

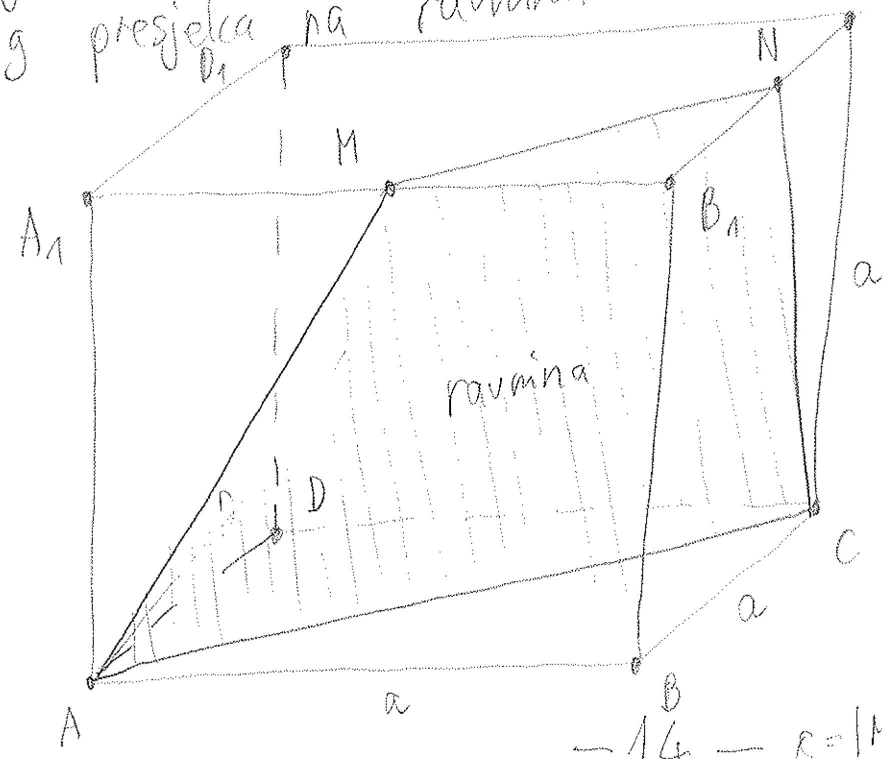
$$\Rightarrow \boxed{\beta = 70{,}529^\circ = 70^\circ 31'}$$

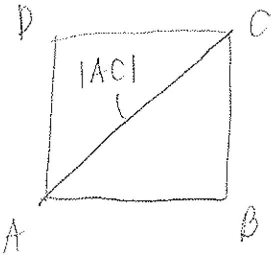
(3) (Primjer 5)

Kocka $ABCDA_1B_1C_1D_1$ duljine brida a presječena je ravninom koja prolazi dijagonalom $\overline{AC}$ osnovke i polovištem M brida A_1B_1 gornje osnovke. Kolika je površina tog presjeka? Kolika je površina ortogonalne projekcije istog presjeka na ravninu ABC?

Duljine osnovica

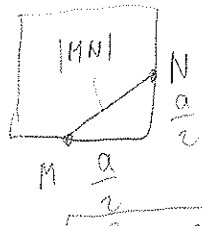
$|AC| = ?$





$a = |AC| = \sqrt{a^2 + a^2} = \sqrt{2}a$

$|MN| = ?$



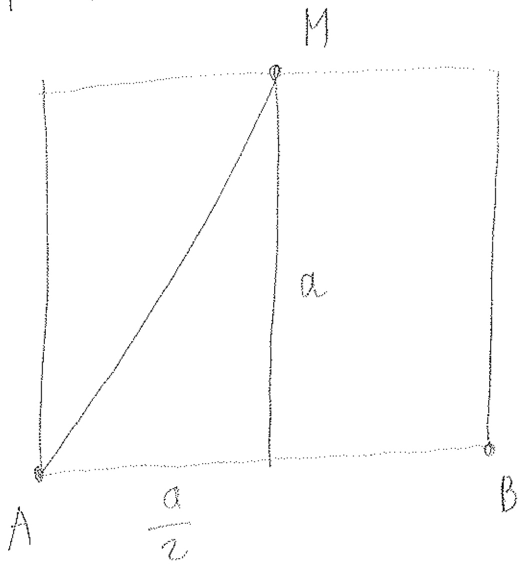
$c = |MN| = \sqrt{\frac{a^2}{4} + \frac{a^2}{4}} = \frac{\sqrt{2}}{2}a$

Bočni bridovi

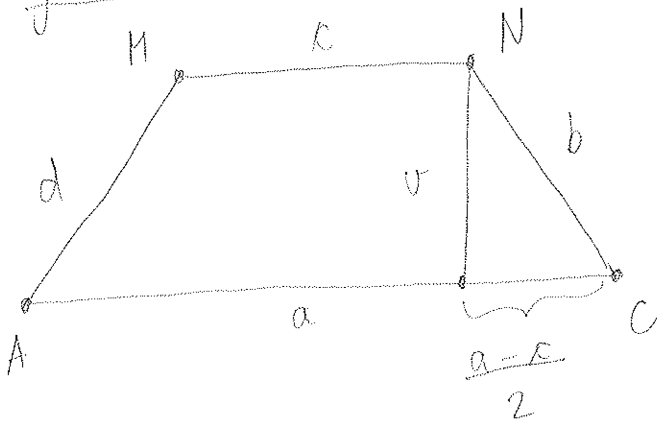
$|AM| = |CN|$

$$d = |AM| = \sqrt{a^2 + \frac{a^2}{4}} = \frac{\sqrt{5}}{2}a$$

$$b = d = \frac{\sqrt{5}}{2}a$$



jednakokračni trapez



$$p = \frac{a+c}{2} \cdot v$$

$$v = \sqrt{b^2 - \left(\frac{a-c}{2}\right)^2}$$

$$\left(\frac{a-c}{2}\right) = \frac{\sqrt{2}a - \frac{\sqrt{2}}{2}a}{2} = \frac{\sqrt{2}}{4}a$$

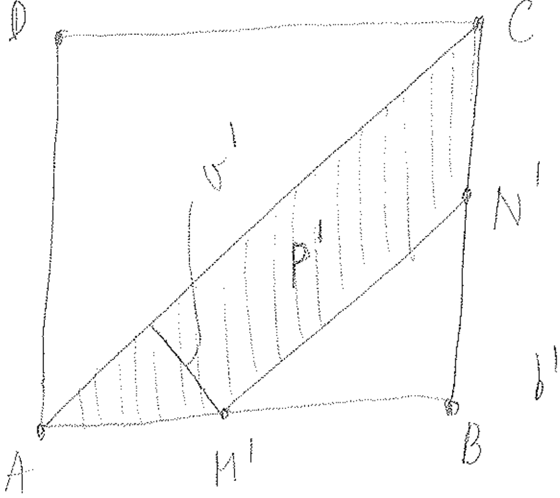
$$v = \sqrt{\left(\frac{\sqrt{5}}{2}a\right)^2 - \left(\frac{\sqrt{2}}{4}a\right)^2} = \sqrt{\frac{5}{4}a^2 - \frac{1}{8}a^2} = \sqrt{\frac{9}{8}}\,a = \frac{3}{2\sqrt{2}}a \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$v = \frac{3\sqrt{2}}{4}a$$

$$p = \frac{\sqrt{2}a + \frac{\sqrt{2}}{2}a}{2} \cdot \frac{3\sqrt{2}}{4}a = \frac{3}{2}\sqrt{2}a \cdot \frac{1}{2} \cdot \frac{3\sqrt{2}}{4} = \left(\frac{3\sqrt{2}}{4}\right)^2 a^2$$

$$\boxed{p = \frac{9 \cdot 2}{16}a^2 = \frac{18}{16}a^2 = \frac{9}{8}a^2}$$

Površina ortogonalne projekcije na ravninu ABC :



$c' = |M'N'| = |MN| = \frac{\sqrt{2}}{2} a$

$a = |AC| = \sqrt{2}\, a$

$b' = d' = |AM'| = |CN'| = \frac{a}{2}$

$$v' = \sqrt{\left(\frac{a}{2}\right)^2 - \left(\frac{\sqrt{2}}{4} a\right)^2} = \sqrt{\frac{a^2}{4} - \frac{1}{8} a^2} = \frac{1}{2\sqrt{2}} a \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$v' = \frac{\sqrt{2}}{4} a$$

$$p' = \frac{a + c'}{2} \cdot v' = \frac{\sqrt{2} a + \frac{\sqrt{2}}{2} a}{2} \cdot \frac{\sqrt{2}}{4} a$$

$$\boxed{p' = \frac{3}{4} \sqrt{2}\, a^2 \cdot \frac{\sqrt{2}}{4} = \frac{3 \cdot \cancel{2}^{1}}{\cancel{16}_{8}} a^2 = \frac{3}{8} a^2}$$

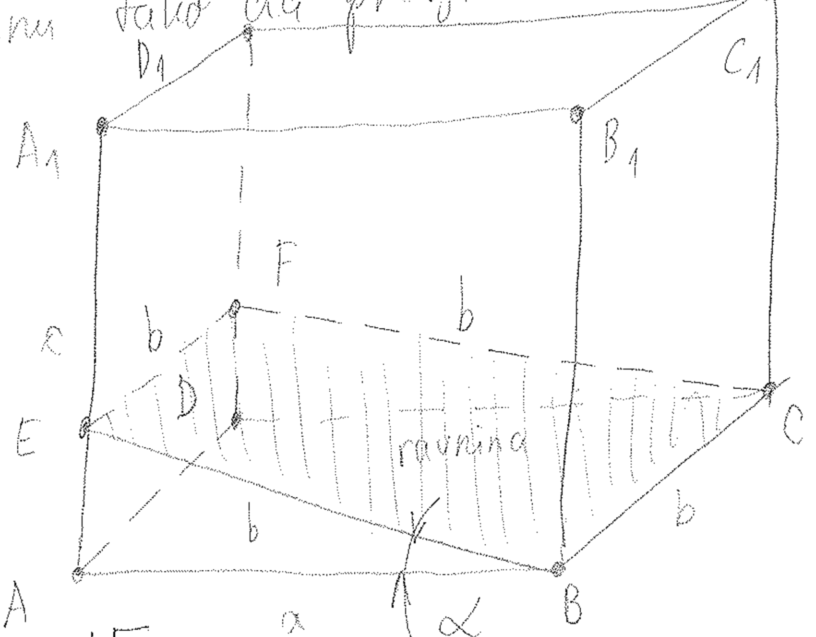
4 (Primjer 7)
Bridovi osnovke kvadra dugi su 4 cm i 5 cm, duljina bočnog brida je 6 cm. Pod kojim kutem prema ravnini osnovke treba postaviti ravninu tako da presjek kvadra tom ravninom bude kvadrat?

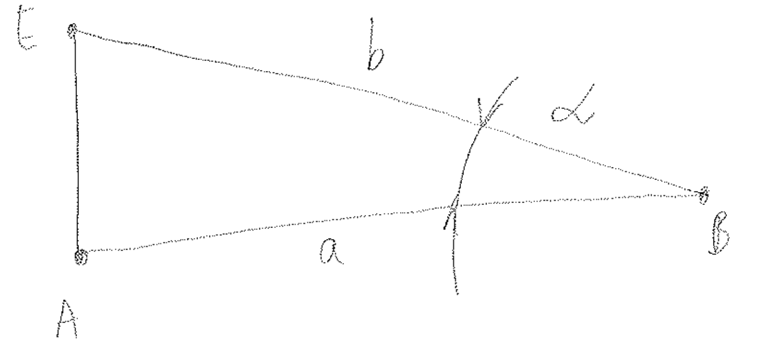
$a = 4$ cm

$b = 5$ cm

$c = 6$ cm

$\alpha = ?$





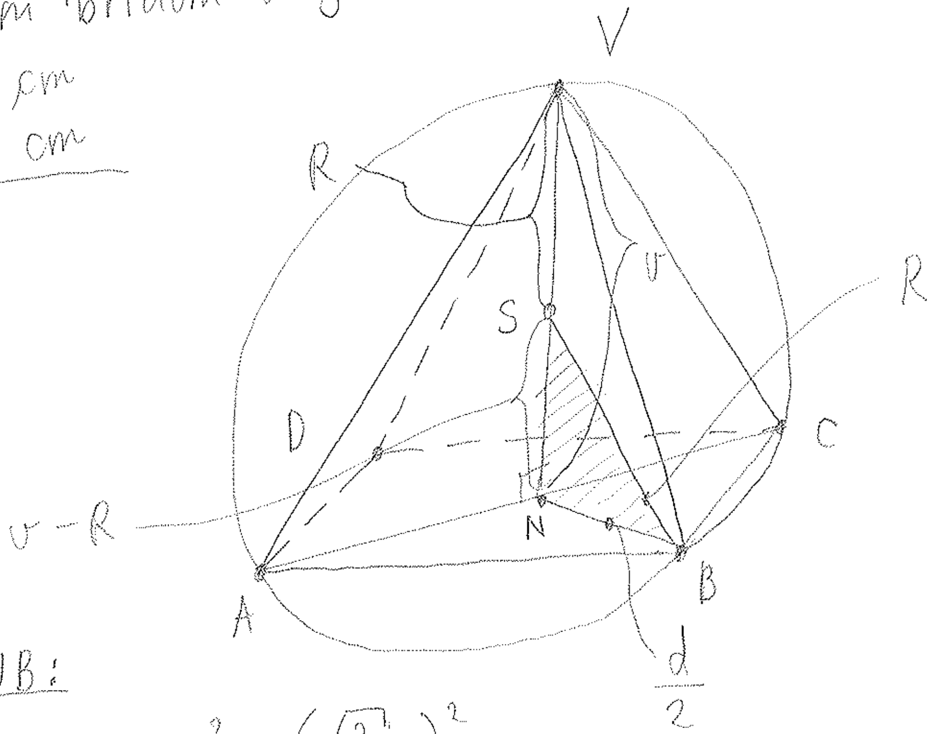
$\cos\alpha = \frac{a}{b} = \frac{4}{5} = 0{,}8$

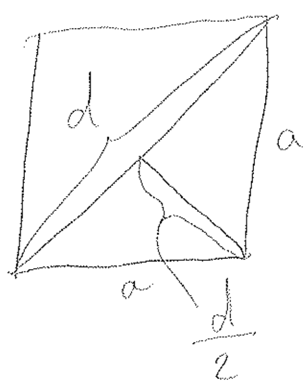
$\Rightarrow \boxed{\alpha = 36{,}869^\circ = 36^\circ 52' 11''}$

⑤ (Primjer 8)
Koliki je promjer sfere opisane četverostranoj piramidi s osnovnim bridom duljine 8 cm i visinom duljine 10 cm?

$a = 8$ cm
$v = 10$ cm

$R = ?$





trokut SNB:

$$R^2 = (v-R)^2 + \left(\frac{\sqrt{2}}{2}a\right)^2$$

$$\cancel{R^2} = v^2 - 2vR + \cancel{R^2} + \frac{1}{2}a^2$$

$$2vR = v^2 + \frac{1}{2}a^2$$

$$\boxed{R = \frac{v^2 + \frac{1}{2}a^2}{2v}} = \frac{10^2 + \frac{1}{2}\cdot 8^2}{2\cdot 10} = \frac{100+32}{20} = \boxed{6{,}6 \text{ cm}}$$

$d = \sqrt{a^2 + a^2}$

$d = \sqrt{2}\,a$

02
zadaci – 3, 5, 10, 12, 13
ispit 1 – 5, 7, 9, 10
ispit 2 – 6, 9, 10