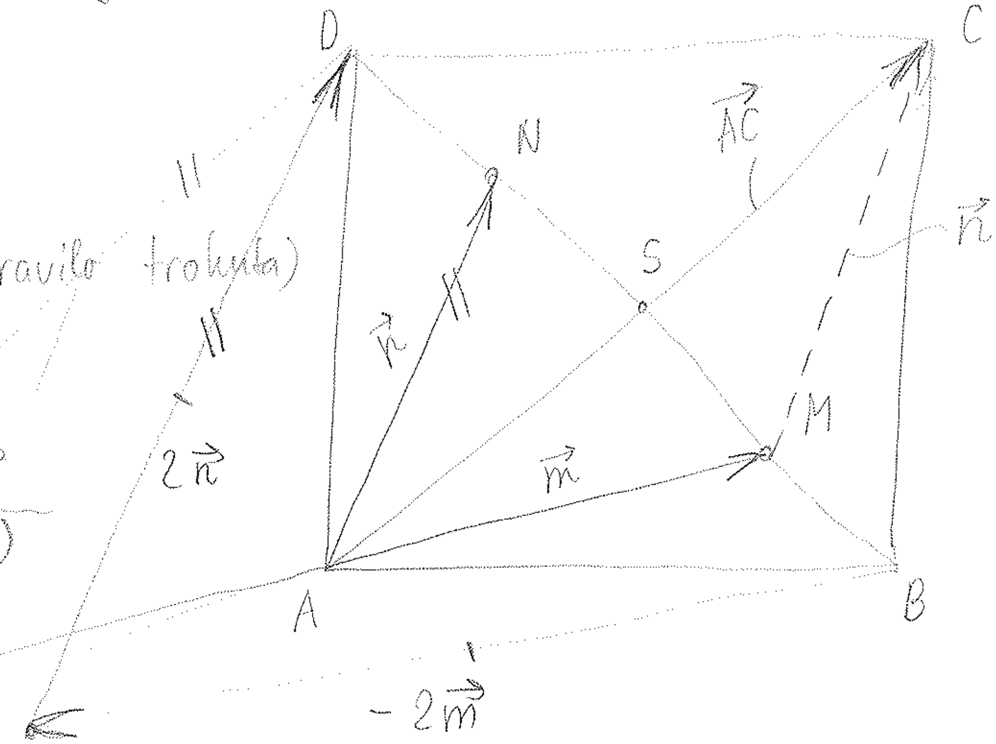
21. VEKTORI

① (Primjer 2)

Dan je kvadrat $ABCD$, točka S sjecište je njegovih dijagonala, a točke M i N polovišta su dužina $\overline{BS}$, odnosno $\overline{DS}$. Izrazimo vektore $\overrightarrow{AC}$, $\overrightarrow{BD}$, $\overrightarrow{AB}$ i $\overrightarrow{BC}$ kao linearne kombinacije vektora $\overrightarrow{AM} = \vec{m}$ i $\overrightarrow{AN} = \vec{n}$.

$\overrightarrow{AM} = \vec{m}$

$\overrightarrow{AN} = \vec{n}$



$\overrightarrow{AC} = \vec{m} + \vec{n}$ (pravilo trokuta)

$\overrightarrow{BD} = ?$

$\overrightarrow{BD} = -2\vec{m} + 2\vec{n}$ (riješeno na kraju!!!)

$$\overrightarrow{AB} = \overrightarrow{AM} + \overrightarrow{MB} = \vec{m} - \frac{1}{4}\overrightarrow{BD} = \vec{m} - \frac{1}{\cancel{4}_{2}}\left(-\cancel{2}^{1}\vec{m} + \cancel{2}^{1}\vec{n}\right)$$

$$\overrightarrow{AB} = \vec{m} + \frac{1}{2}\vec{m} - \frac{1}{2}\vec{n} = \frac{3}{2}\vec{m} - \frac{1}{2}\vec{n}$$

$$\overrightarrow{BC} = -\overrightarrow{AB} + \overrightarrow{AC} = -\left(\frac{3}{2}\vec{m} - \frac{1}{2}\vec{n}\right) + \vec{m} + \vec{n} = -\frac{1}{2}\vec{m} + \frac{3}{2}\vec{n}$$

$$\overrightarrow{BD} = -\overrightarrow{AB} + \overrightarrow{BC} = \underbrace{-\frac{3}{2}\vec{m} + \frac{1}{2}\vec{n}}_{-\overrightarrow{AB}}\ \underbrace{-\frac{1}{2}\vec{m} + \frac{3}{2}\vec{n}}_{\overrightarrow{BC}}$$

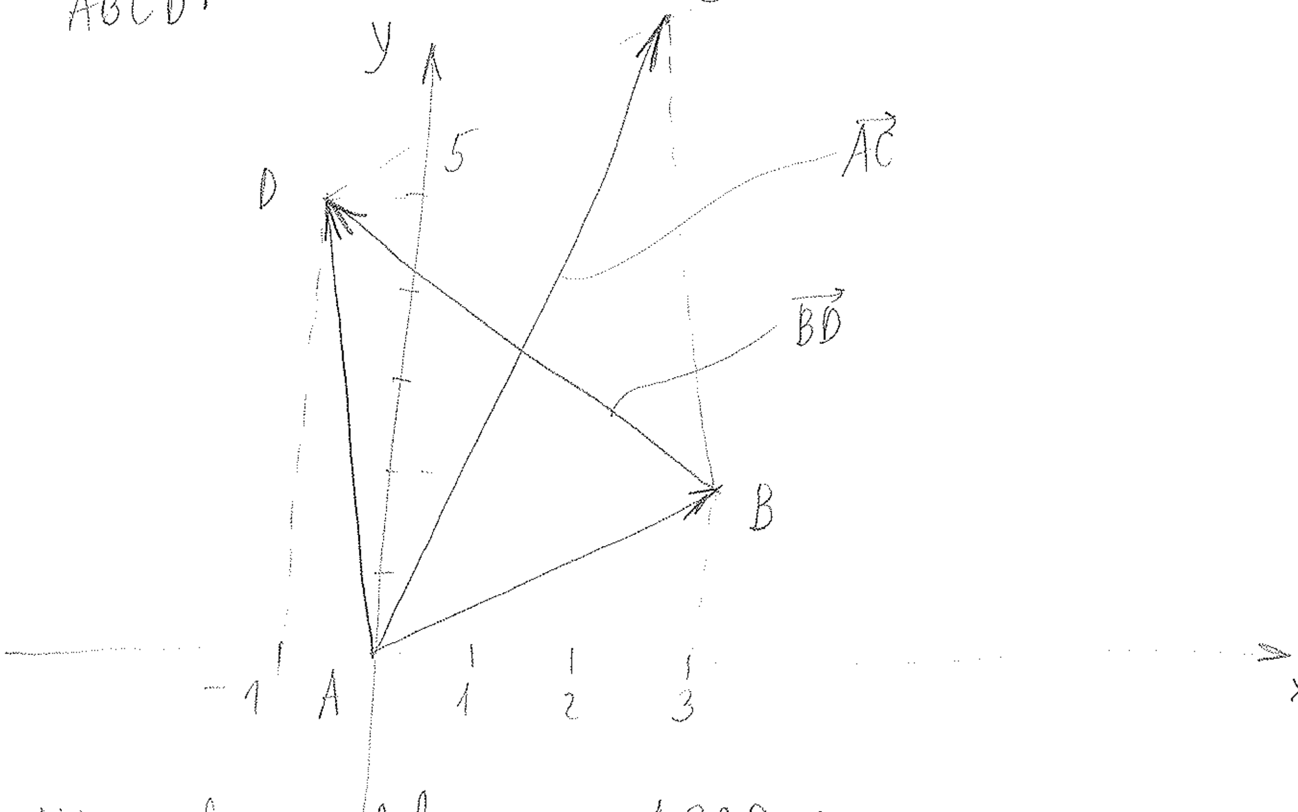
$$\overrightarrow{BD} = -2\vec{m} + 2\vec{n}$$

② (Primjer 3)

Vektorima $\overrightarrow{AB} = 3\vec{i} + 2\vec{j}$ i $\overrightarrow{AD} = -\vec{i} + 5\vec{j}$ odreden je paralelogram ABCD.

1) Kolike su duljine dijagonala paralelograma ABCD?

2) Odredimo kut izmedu dijagonala paralelograma ABCD.



1) Dijagonale paralelograma ABCD:

$\overrightarrow{AC} = ?$

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD} = 3\vec{i} + 2\vec{j} + (-\vec{i} + 5\vec{j}) = 2\vec{i} + 7\vec{j}$$

$$|\overrightarrow{AC}| = \sqrt{2^2 + 7^2} = \sqrt{4+49} = \sqrt{53}$$

$$\overrightarrow{BD} = -\overrightarrow{AB} + \overrightarrow{AD} = -(3\vec{i} + 2\vec{j}) + (-\vec{i} + 5\vec{j})$$

$$\overrightarrow{BD} = -4\vec{i} + 3\vec{j}$$

$$|\overrightarrow{BD}| = \sqrt{(-4)^2 + 3^2} = 5$$

2) Kut izmedu dijagonala paralelograma:

skalarni umnožak vektora:

$$\overrightarrow{AC} \cdot \overrightarrow{BD} = |\overrightarrow{AC}| \cdot |\overrightarrow{BD}| \cdot \cos\varphi \Rightarrow \cos\varphi = \frac{\overrightarrow{AC} \cdot \overrightarrow{BD}}{|\overrightarrow{AC}| \cdot |\overrightarrow{BD}|} \quad (*)$$

$$\cos\varphi = \frac{2\cdot(-4) + 7\cdot 3}{\sqrt{53}\cdot 5} = \frac{-8+21}{5\sqrt{53}} = \frac{13}{5\sqrt{53}} \Rightarrow \boxed{\varphi = 69^\circ 4' 32''}$$

FSB cad lab
Katedra za konstruiranje i razvoj proizvoda

(Pr 4)

(3) Dani su vektori $\vec{a} = -\vec{i} - \vec{j}$, $\vec{b} = \vec{i} + 3\vec{j}$, $\vec{c} = -2\vec{i} + 4\vec{j}$.
Odredite vektor $\vec{v}$ kolinearan s vektorom $\vec{a}$ i duljine jednake duljini vektora $\vec{b} + \vec{c}$.

$$\left.\begin{array}{l} \vec{a} = -\vec{i} - \vec{j} \\ \vec{b} = \vec{i} + 3\vec{j} \\ \vec{c} = -2\vec{i} + 4\vec{j} \end{array}\right\} \vec{b} + \vec{c} = (\vec{i} + 3\vec{j}) + (-2\vec{i} + 4\vec{j}) = \vec{d}$$

$$= -\vec{i} + 7\vec{j}$$

duljina vektora:

$$|\vec{b} + \vec{c}| = \sqrt{d_x^2 + d_y^2}$$

$$= \sqrt{(-1)^2 + 7^2} = \sqrt{50} = 5\sqrt{2}$$

kolinearnost vektora $\vec{v} = k \cdot \vec{a}$, $k \neq 0$, $k \in \mathbb{R}$

$$|\vec{v}| = |\vec{b} + \vec{c}|$$

$$k \cdot \vec{a} = k(-\vec{i} - \vec{j}) = -k\vec{i} - k\vec{j}$$

$$|\vec{v}| = \sqrt{(-k)^2 + (-k)^2} = \sqrt{2k^2} = |k|\sqrt{2}$$

$$|k|\sqrt{2} = 5\sqrt{2}$$

$$k = \pm 5$$

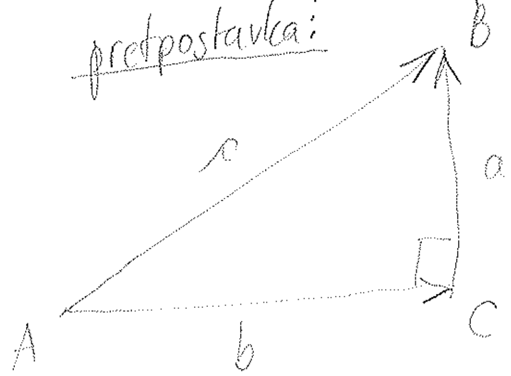
$$\boxed{\vec{v} = \pm 5(-\vec{i} - \vec{j})}$$

③ (Primjer 5)

Ako su A, B, C vrhovi trokuta, te $\vec{AB} = \vec{i} - 3\vec{j}$, $\vec{AC} = 2\vec{i} + k\vec{j}$, odredimo realni broj $k \neq 0$, tako da trokut $\triangle ABC$ bude pravokutan s pravim kutem pri vrhu C.

$$\vec{AB} = \vec{i} - 3\vec{j}$$
$$\vec{AC} = 2\vec{i} + k\vec{j}$$

pretpostavka:



$$\vec{BC} = \vec{AB} - \vec{AC}$$
$$\vec{BC} = (\vec{i} - 3\vec{j}) - (2\vec{i} + k\vec{j})$$
$$\vec{BC} = -\vec{i} - (3+k)\vec{j}$$

kako je $\triangle ABC$ pravokutan, tada mora biti:

$$\vec{AC} \cdot \vec{BC} = 0$$
$$(2\vec{i} + k\vec{j}) \cdot (-\vec{i} - (3+k)\vec{j}) = 0$$
$$2 \cdot (-1) + (-k) \cdot (3+k) = 0$$
$$-2 - 3k - k^2 = 0$$
$$k^2 + 3k + 2 = 0 \quad \begin{array}{l} \nearrow k_1 = -2 \\ \searrow k_2 = -1 \end{array}$$

2 rješenja:

I) $\vec{BC} = -\vec{i} - \vec{j}$

II) $\vec{BC} = -\vec{i} - 2\vec{j}$

II način: okrenuti trokut prema dolje, tj. $\vec{AB}$ u drugom smjeru

(4) (zadaci – 11)

Dani su vektori $\vec{a} = -2\vec{i} + \vec{j}$, $\vec{b} = 3\vec{i} - 2\vec{j}$. Odredite vektor $\vec{c}$ za kojega je $\vec{a} \cdot \vec{c} = 3$, te $\vec{b} \cdot \vec{c} = -5$.

$$\vec{a} \cdot \vec{c} = 3 = |\vec{a}| \cdot |\vec{c}| \cdot \cos \varphi_1$$

$$|\vec{a}| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$$

$$|\vec{c}| = \sqrt{x_c^2 + y_c^2}$$

$$\sqrt{5} \cdot \sqrt{x_c^2 + y_c^2} \cdot \cos \varphi_1 = 3$$

NIJE DOBRO!

$$\vec{b} \cdot \vec{c} = -5 = |\vec{b}| \cdot |\vec{c}| \cdot \cos \varphi_2$$

$$|\vec{b}| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$$

$$|\vec{c}| = \sqrt{x_c^2 + y_c^2}$$

$$\sqrt{13} \cdot \sqrt{x_c^2 + y_c^2} \cdot \cos \varphi_2 = -5$$

$$\vec{a} \cdot \vec{c} = a_x c_x + a_y c_y = -2c_x + c_y = 3 \quad / \cdot 2$$

$$\vec{b} \cdot \vec{c} = b_x c_x + b_y c_y = 3c_x - 2c_y = -5$$

$$\left.\begin{aligned} -4c_x + 2c_y &= 6 \\ 3c_x - 2c_y &= -5 \end{aligned}\right\} (+)$$

$$-c_x = 1$$

$$c_x = -1$$

$$2 + c_y = 3$$

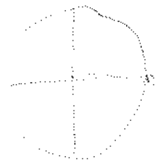
$$c_y = 1$$

$$\boxed{\vec{c} = -\vec{i} + \vec{j}}$$

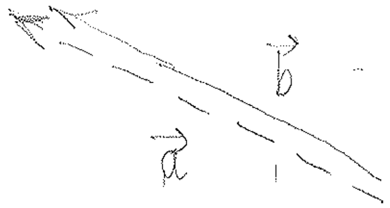
(5) (zadaci – 13)

Dan je vektor $\vec{b} = -2\vec{i} + \vec{j}$. Odredimo vektor $\vec{a}$ istog smjera i iste orijentacije, kao i vektor $\vec{b}$ i duljine $3\sqrt{5}$.

isti smjer (kolinearni), ista orijentacija – leže na paralelnim pravcima i gledaju u istu stranu



$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \underbrace{\cos 0^\circ}_{=1}$$



$$|\vec{b}| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$$

$$a_x b_x + a_y b_y = \sqrt{5} \cdot 3\sqrt{5}$$

$$\sqrt{a_x^2 + a_y^2} = 3\sqrt{5} \quad /^2$$

$$-2a_x + a_y = 15 \Rightarrow a_y = 15 + 2a_x$$

$$a_x^2 + a_y^2 = 45$$

$$a_x^2 + (15 + 2a_x)^2 = 45$$

$$a_x^2 + 225 + 60a_x + 4a_x^2 = 45$$

$$5a_x^2 + 60a_x + 180 = 0 \quad /:5$$

$$a_x^2 + 12a_x + 36 = 0$$

$$a_{x_{1,2}} = \frac{-12 \pm \sqrt{144 - 144}}{2} = -6$$

$$a_y = 15 + 2 \cdot (-6) = 3$$

$$\boxed{\vec{a} = -6\vec{i} + 3\vec{j}}$$

(6) (zadaci – 16)
Odredite jedinični vektor okomit na vektor $\overrightarrow{AB}$, $A(-1, 2)$, $B(3, -1)$.

$$\overrightarrow{AB} = \sqrt{(3-(-1))^2}\,\vec{i} + \sqrt{(-1-2)^2}\,\vec{j} = 4\vec{i} + 3\vec{j} = \vec{a}$$

$$\vec{e} = e_x \vec{i} + e_y \vec{j}$$

$$|\vec{e}| = 1 = \sqrt{e_x^2 + e_y^2} \quad (1)$$

$$\overrightarrow{AB} \perp \vec{e} \Rightarrow \overrightarrow{AB} \cdot \vec{e} = 0$$

$$a_x e_x + a_y e_y = 0$$

$$4 e_x + 3 e_y = 0 \quad (2)$$

$$(1)/^2 \Rightarrow e_x^2 + e_y^2 = 1$$

$$\text{iz } (2): \Rightarrow \underbrace{e_x = -\frac{3}{4} e_y}$$

$$\left(-\frac{3}{4} e_y\right)^2 + e_y^2 = 1$$

$$\frac{9}{16} e_y^2 + e_y^2 = 1$$

$$\frac{25}{16} e_y^2 = 1 \quad / \cdot \frac{16}{25}$$

$$e_y^2 = \frac{16}{25}$$

$$\underbrace{e_y = \frac{4}{5}}$$

$$e_x = -\frac{3}{4} \cdot \frac{4}{5} = -\frac{3}{5}$$

$$\boxed{\vec{e} = -\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}}$$

⑦ (ispit 1 − 6)

Vektori $\vec{AB} = -\vec{i} + 2\vec{j}$ i $\vec{CD} = 3\vec{i} - \vec{j}$ zatvaraju kut:

$$\vec{AB} = \vec{a} = -\vec{i} + 2\vec{j}$$

$$\vec{CD} = \vec{b} = 3\vec{i} - \vec{j}$$

$$\cos\varphi = \frac{a_x b_x + a_y b_y}{\sqrt{a_x^2 + a_y^2} \cdot \sqrt{b_x^2 + b_y^2}} = \frac{-1 \cdot 3 + 2 \cdot (-1)}{\sqrt{(-1)^2 + 2^2} \cdot \sqrt{3^2 + (-1)^2}}$$

$$\cos\varphi = \frac{-5}{\sqrt{5} \cdot \sqrt{10}} = \frac{-5}{\sqrt{50}} = \frac{-\cancel{5}^{1}}{\cancel{5}_{1}\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$\Rightarrow \boxed{\varphi = 135^\circ = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}}$$

(B)

$$6 \cdot (x - 4) + 3 \cdot 7 = 0$$

$$6x - 24 + 21 = 0$$

$$6x - 3 = 0$$

$$x = \frac{1}{2}$$

(D)

DZ

zadaci − 1, 3, 6, 8, 17

ispit 1 − 2, 7, 8

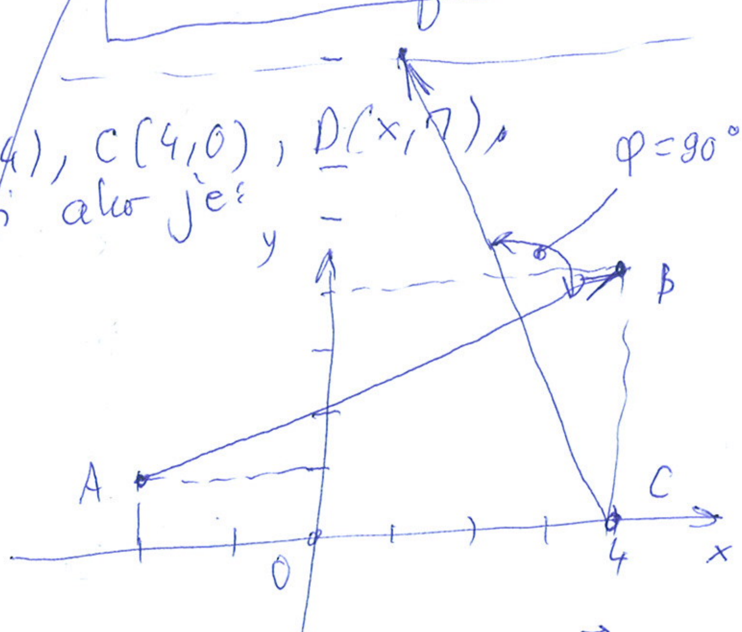
ispit 2 − 3, 6, 9, 10

⑧ (ispit 1 − 8)

Dane su tačke $A(-2,1)$, $B(4,4)$, $C(4,0)$, $D(x,7)$.

Vektori $\vec{AB}$ i $\vec{CD}$ su okomiti ako je:

$x = ?$



okomitost vektora

kut između vektora

$\cos 90^\circ = 0$

$$a_x b_x + a_y b_y = 0$$

$$\cos\varphi = \frac{a_x b_x + a_y b_y}{\sqrt{a_x^2 + a_y^2} \cdot \sqrt{b_x^2 + b_y^2}} = 0$$

$a \ldots \vec{AB}$ $\quad (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} = 0$

$b \ldots$

$(x_D - x_C)\vec{i} + (y_D - y_C)\vec{j} = 0$ $\quad (4 - (-2))\vec{i} + (4 - 1)\vec{j} = 0$

$(x - 4)\vec{i} + (7 - 0)\vec{j} = 0$

$-4 - (x-4)\vec{i} + 7\vec{j} = 0$ $\quad 6\vec{i} + 3\vec{i} = 0$

22. ARITMETIČKI I GEOMETRIJSKI NIZ

① (Primjer 2)
U nizu (a_n) ako je $a_1 = 1$, $a_2 = 2$, a za $n \geq 2$ je
$a_{n+2} = \frac{a_{n+1}}{a_n}$ odredimo a_{777}.

članovi niza: $\underbrace{1, 2, 2, 1, \frac{1}{2}, \frac{1}{2}}_{}, \ldots$ ponavlja se

777 član $\longrightarrow$ $\frac{777}{6} = 129$ ostatak 3

$$\boxed{a_{777} = 2}$$

② (Primjer 5)
U aritmetičkom je nizu $a_5 + a_9 = 11$, $a_5 \cdot a_9 = 28$. Odredimo opći član ovog niza.

opći član aritmetičkog niza: $a_n = a_1 + (n-1)d$

$\downarrow$

$a_5 = a_1 + (5-1)d = a_1 + 4d$

$a_9 = a_1 + (9-1)d = a_1 + 8d$

$a_1 + 4d + a_1 + 8d = 11 \quad (1)$

$(a_1 + 4d)(a_1 + 8d) = 28 \quad (2)$

$2a_1 + 12d = 11 \Rightarrow 2a_1 = 11 - 12d \;/:2$

$a_1 = \frac{11}{2} - 6d \rightarrow (2)$

$\left(\frac{11}{2} - 6d + 4d\right)\left(\frac{11}{2} - 6d + 8d\right) = 28$

$$\left(\frac{11}{2}-2d\right)\left(\frac{11}{2}+2d\right)=28$$

$$\left(\frac{11}{2}\right)^2-(2d)^2=28$$

$$-4d^2=28-\left(\frac{11}{2}\right)^2=28-\frac{121}{4}=\frac{112-121}{4}=-\frac{\ }{4}$$

$$d^2=\frac{9}{16}\Rightarrow d=\frac{3}{4}$$

$$a_1=\frac{11}{2}-6\cdot\frac{3}{4}=\frac{11}{2}-\frac{9}{2}=\boxed{1}$$

$$a_5=1+4\cdot\frac{3}{4}=4$$

$$a_9=1+8\cdot\frac{3}{4}=7$$

$$\boxed{a_n=1+\frac{3}{4}(n-1)}$$

③ (Primjer 6)
Odredimo zbroj svih troznamenkastih brojeva koji pri dijeljenju s 5 imaju ostatak 2 ili 3.
Gledamo prvih nekoliko troznamenkastih brojeva koji pri dijeljenju s 5 daju ostatak 2 ili 3:

102, 103, 107, 108, 112, 113, 117, 118 ....

dva podniza

1. podniz → 102, 107, 112, 117, ... 997 → broj članova

2. podniz → 103, 108, 113, 118, ... 998

$$\frac{997-102}{5}+1=\underline{180}$$

Tražena suma članova niza:

općenito: $S_n = \frac{n}{2}(a_1 + a_n)$

$$S = S_1 + S_2 = \frac{n_1}{2}(a_{11} + a_{n1}) + \frac{n_2}{2}(a_{12} + a_{n2})$$

$$= \frac{180}{2}\left[(102 + 997) + (103 + 998)\right]$$

$$\boxed{S = 90 \cdot [1099 + 1101] = 198\,000}$$

④ Nađimo zbroj svih negativnih članova niza čiji je
(Pr 8) opći član: $a_n = \frac{1}{2}n - 25$.

pošto se radi o negativnim članovima:

$$a_n < 0$$

$$\frac{1}{2}n - 25 < 0 \Rightarrow n < 50$$

$$n = 49$$

za $n = 1 \Rightarrow a_1 = \frac{1}{2} - 25 = -\frac{49}{2}$

za $n = 49 \Rightarrow a_{49} = \frac{49}{2} - 25 = -\frac{1}{2}$

$$\boxed{S = \frac{49}{2}\left(-\frac{49}{2} - \frac{1}{2}\right) = -612{,}5}$$

⑤ (Primjer 11)
Drugi član geometrijskog niza jednak je 3, a peti član istog niza je 12. Odredimo osmi član ovog niza.

$a_2 = 3$

$a_5 = 12$

$a_8 = ?$

opći član geometrijskog niza:

$$a_n = a_1 \cdot q^{n-1}$$

$$q = \frac{a_n}{a_{n-1}}$$

I) $a_5 = a_2 \cdot q^3$

$a_8 = a_5 \cdot q^3$

$12 = 3 \cdot q^3 \implies q^3 = 4$

$\boxed{a_8 = 12 \cdot 4 = 48}$

II) $a_n^2 = a_{n-k} \cdot a_{n+k}$

$a_5^2 = a_2 \cdot a_8$

$12^2 = 3 \cdot a_8 \implies \boxed{a_8 = \frac{144}{3} = 48}$

6 (Primjer 13)

U nekom je geometrijskom nizu $a_1 \cdot a_3 \cdot a_5 \cdot \ldots \cdot a_{31} = 625$.
Koliki je a_{16}?

$a_m \cdot a_n = a_u \cdot a_v \iff m+n = u+v$

svojstvo geometrijskog niza

$m+n=u+v$

$a_1 \cdot a_{31} = a_3 \cdot a_{29} = \ldots = a_{15} \cdot a_{17} = a_{16}^2$

16 parova

$a_{16}^{2^4} = 625 = 5^4$

ostali parovi:

$a_5 \cdot a_{27}$ $\quad a_{11} \cdot a_{21}$

$a_7 \cdot a_{25}$ $\quad a_{13} \cdot a_{19}$

$a_9 \cdot a_{23}$ $\quad a_{15} \cdot a_{17}$

$a_{16}^{16} = 5^4 \quad / \frac{1}{16}$

$\boxed{a_{16} = \sqrt[4]{5}}$

DZ

zadaci – 1, 5, 9, 12, 15
ispit 1 – 2, 6, 9
ispit 2 – 5, 7, 9, 10
ispit 3 – 4, 6, 9
ispit 4 – 3, 5, 7, 8

23. DERIVACIJE FUNKCIJA

① (Primjer 2)
Derivirajmo funkciju $f(x) = \frac{2}{x} + 3\sqrt{x}$

$$\frac{2}{x} + 3\sqrt{x} = 2x^{-1} + 3 \cdot x^{\frac{1}{2}} \Big/ \frac{d}{dx} \quad \text{ili} \quad '$$

$$\left(2x^{-1} + 3x^{\frac{1}{2}}\right)' = 2 \cdot (-1)\, x^{-1-1} + 3 \cdot \frac{1}{2}\, x^{\frac{1}{2}-1}$$

$$= -2x^{-2} + \frac{3}{2} x^{-\frac{1}{2}}$$

$$\boxed{f'(x) = \frac{-2}{x^2} + \frac{3}{2\sqrt{x}}}$$

② (Primjer 3)
Derivirajmo funkciju 1) $f(x) = (x+1)\sqrt{x} + \frac{1}{x}$
2) $g(x) = x \cdot \sin x$

1) $$f'(x) = \left[(x+1)\sqrt{x} + \frac{1}{x}\right]' = \left[(x+1)'\sqrt{x} + (x+1)(\sqrt{x})' + \left(\frac{1}{x}\right)'\right]$$

$$= \left[1 \cdot \sqrt{x} + (x+1)\frac{1}{2}x^{-\frac{1}{2}} + (-1)\, x^{-2}\right]$$

$$= \sqrt{x} + \frac{1}{2}(x+1)\frac{1}{\sqrt{x}} - \frac{1}{x^2}$$

$$= \sqrt{x} + \frac{x+1}{2\sqrt{x}} - \frac{1}{x^2} = \sqrt{x} + \frac{\sqrt{x}}{2} + \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$$

$$\boxed{f'(x) = \frac{3}{2}\sqrt{x} + \frac{1}{2\sqrt{x}} - \frac{1}{x^2}}$$

2) $\boxed{g'(x) = x' \cdot \sin x + x \cdot (\sin x)' = \sin x + x \cos x}$

(3) (Primjer 4)

Odredimo derivaciju funkcije $f(x) = \sin\sqrt{1+x^2}$,

$$f'(x) = \left(\sin\sqrt{1+x^2}\right)' = \cos\sqrt{1+x^2} \cdot \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x$$

$$\boxed{f'(x) = \frac{\cancel{2}^{1} x \cdot \cos\left(\sqrt{1+x^2}\right)}{\cancel{2}_{1}\sqrt{1+x^2}} = \frac{x \cdot \cos\left(\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}}$$

(4) (Primjer 5)

Derivirajmo funkciju $f(x) = \ln\sqrt{xe^x}$,

$$\boxed{f'(x) = \left(\ln\sqrt{xe^x}\right)' = \frac{1}{\sqrt{xe^x}}\frac{1}{2}(xe^x)^{-\frac{1}{2}} \cdot (e^x + xe^x)}$$

$$= \frac{1}{2xe^x}(e^x + xe^x) = \boxed{\frac{x+1}{2x}}$$

(5) (Primjer 6)

Derivirajmo funkciju $f(x) = \dfrac{\sin x - \cos x}{\sin x + \cos x}$

$$\boxed{f'(x) = \frac{(\sin x - \cos x)'(\sin x + \cos x) - (\sin x - \cos x)(\sin x + \cos x)'}{(\sin x + \cos x)^2}}$$

$$= \frac{(\cos x + \sin x)^2 - (\sin x - \cos x)(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

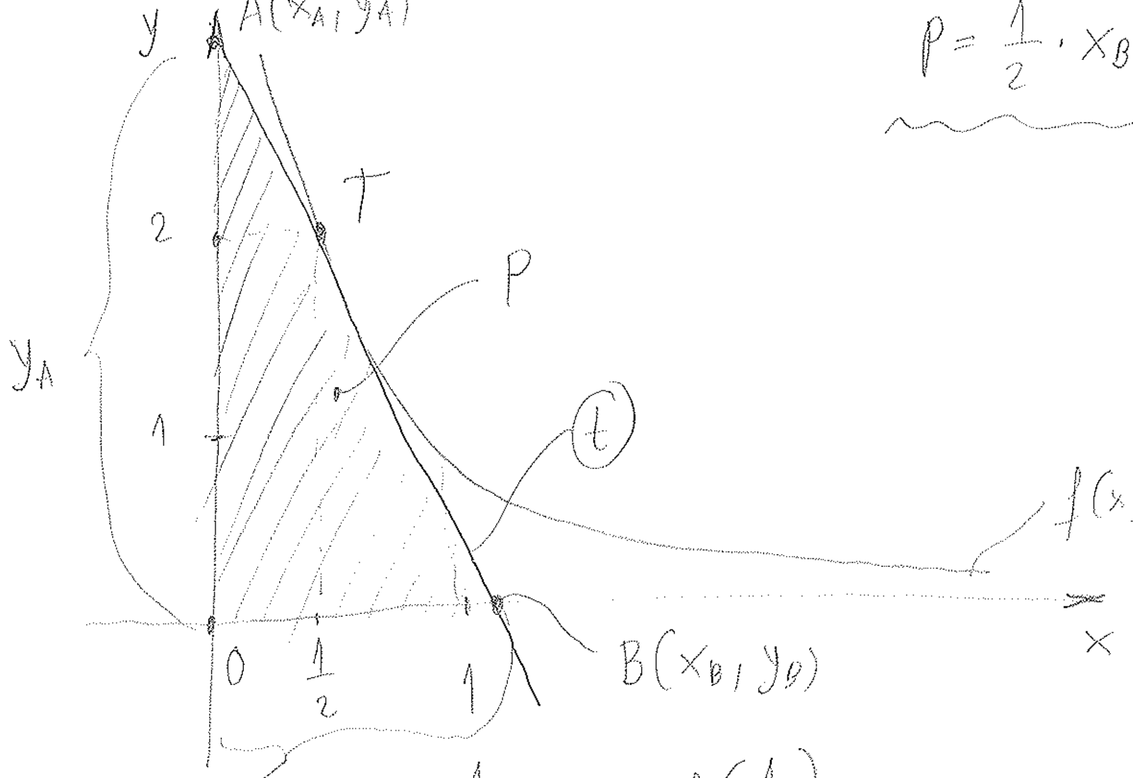
$$= \frac{(\sin x + \cos x)^2 + (\cos x - \sin x)^2}{(\sin x + \cos x)^2} = \frac{2\overbrace{(\sin^2 x + \cos^2 x)}^{=1}}{\underbrace{\sin^2 x + \cos^2 x}_{1} + 2\sin x\cos x}$$

$$= \boxed{\frac{2}{1 + 2\sin x\cos x}}$$

⑥ (Primjer 7)

Kolika je površina trokuta što ga s koordinatnim osima zatvara tangenta na graf funkcije $f(x) = \frac{1}{x}$ u točki s apscisom $x_0 = \frac{1}{2}$?

$$P = \frac{1}{2} \cdot x_B \cdot y_A$$



za $x_0 = \frac{1}{2} \Rightarrow f\left(\frac{1}{2}\right) = 2$

$x_0 = \frac{1}{2}$

$y_0 = 2$

jednadžba tangente: u točki x_0, y_0:

$$y - y_0 = f'(x_0)(x - x_0)$$

$$f'(x) = (x^{-1})' = -x^{-2} = -\frac{1}{x^2}$$

$$f\left(x_0 = \frac{1}{2}\right) = -\frac{1}{\left(\frac{1}{2}\right)^2} = -4$$

$$y - 2 = -4\left(x - \frac{1}{2}\right)$$

$$y - 2 = -4x + 2$$

$$y = -4x + 4 \quad \dots \quad t$$

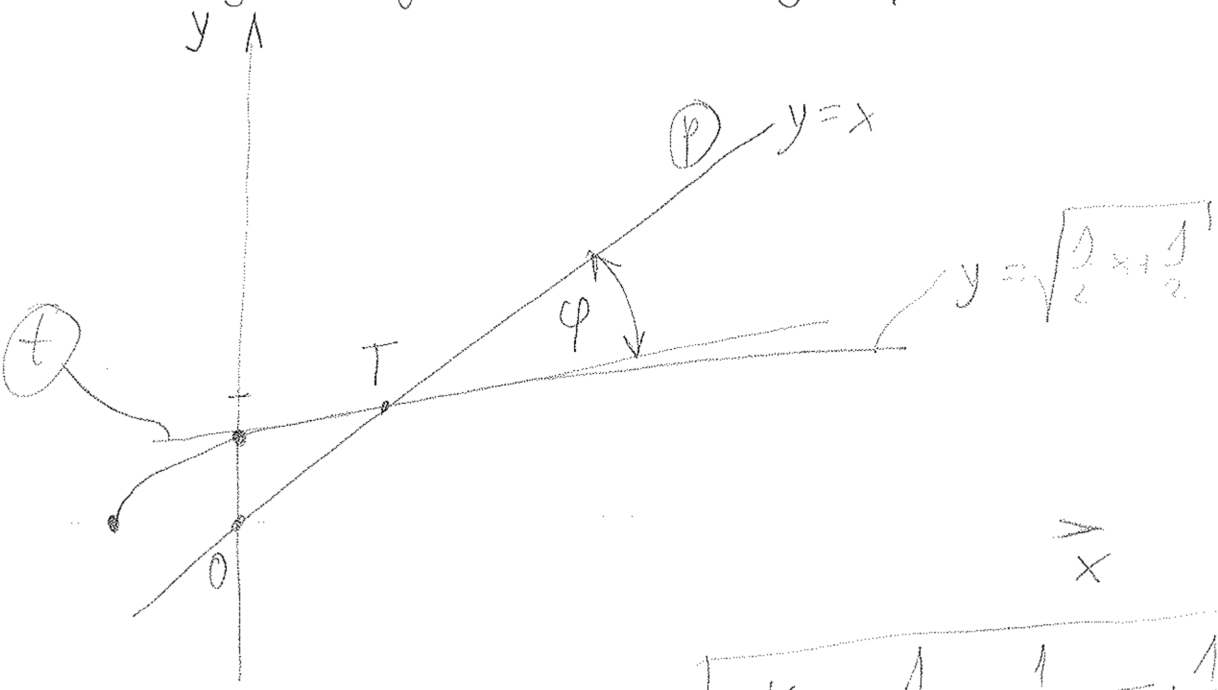
za $x = 0 \Rightarrow y_A = 4$

za $y = 0 \Rightarrow x_B = 1$

$$\boxed{P = \frac{1}{2} \cdot 1 \cdot 4 = 2}$$

⑦ (Primjer 8)

Pod kojim kutem $y = x$ siječe krivulju $y = \sqrt{\frac{1}{2}x + \frac{1}{2}}$?



$$y = \sqrt{\frac{1}{2}x + \frac{1}{2}}$$

$$f'(x) = \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{1}{2}x + \frac{1}{2}}} \cdot \frac{1}{2}$$

$$= \frac{1}{4 \cdot \sqrt{\frac{1}{2}x + \frac{1}{2}}}$$

za $x = 0 \Rightarrow y = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} \approx 0{,}71$

za $y = 0 \Rightarrow 0 = \sqrt{\frac{1}{2}x + \frac{1}{2}} \; \Big/^2$

$$0 = \frac{1}{2}x + \frac{1}{2} \Rightarrow x = -1$$

točka T:

$$x = \sqrt{\frac{1}{2}x + \frac{1}{2}} \; \Big/^2$$

$$x^2 - \frac{1}{2}x - \frac{1}{2} = 0 \; \Big/ \cdot 2$$

$$2x^2 - x - 1 = 0 \Rightarrow x_{1,2} = \frac{1 \pm \sqrt{1 + 8}}{4}$$

$x_1 =$ ...

$x_2 =$ ...

za $x = 1 \Rightarrow y = 1 \quad T(1,1)$

$$p \ldots \left.\begin{array}{l} y = x \\ y' = 1 \end{array}\right\} k_p = 1$$

$t \ldots \quad y - y_T = f'(x)(x - x_T) \qquad f'(x_T) = \frac{1}{4\sqrt{\frac{1}{2} + \frac{1}{2}}} = \frac{1}{4}$

Kut između pravaca:

$$tg\varphi = \left|\frac{k_1-k_2}{1+k_1k_2}\right| = \left|\frac{1-\frac{1}{4}}{1+1\cdot\frac{1}{4}}\right| = \frac{\frac{3}{4}}{\frac{5}{4}} = \frac{3}{5} = 0{,}6$$

$$\Rightarrow \boxed{\varphi = 30{,}964^\circ = 30^\circ 57' 49''}$$

8 (zadaci - 14)

Odredite drugu derivaciju funkcije:

$$f(x) = x^2 \ln x + \sin 2x$$

$$f'(x) = 2x\ln x + x^2\cdot\frac{1}{x} + 2\cos 2x = 2x\ln x + x + 2\cos 2x$$

$$\boxed{f''(x) = 2\ln x + 2\cancel{x}\cdot\frac{1}{\cancel{x}} + 1 - 4\sin 2x = 2\ln x + 3 - 4\sin 2x}$$

9 (zadaci - 24)

Tangenta na graf funkcije $f(x) = x^3+3x^2+9x-12$ paralelna je s pravcem $y=6x$. Napišite jednadžbu te tangente.

$$f(x) = x^3+3x^2+9x-12$$

$p \dots y = 6x$

$t \parallel p \Rightarrow k_t = k_p = 6$

$$f'(x) = k_t$$

$$f'(x) = 3x^2+6x+9 = 6$$

$$3x^2+6x+3 = 0 \quad /:3$$

$$x^2+2x+1=0$$

$$(x+1)^2 = 0 \Rightarrow x = -1$$

$t \dots y_t = 6x + b_t$

za $x=-1 \Rightarrow f(-1) = (-1)^3+3(-1)^2+9(-1)-12 = -19$

za $(-1,-19)$

$$-19 = 6\cdot(-1) + b_t$$

$$-19+6 = b_t$$

$$b_t = -13$$

$t \dots \boxed{y_t = 6x-13}$

(10) (ispit 1 – 5)
Tangente na graf funkcija $f(x) = 2x^2 - 3$ i $g(x) = \ln(3x)$ položene u točkama s istom apscisom su paralelne. Kolika je ta apscisa?

$t_1 \dots f(x) \qquad f'(x) = 4x = k_{t_1}$

$t_2 \dots g(x) \qquad g'(x) = \frac{3}{3x} = \frac{1}{x} = k_{t_2}$

$t_1 \parallel t_2 \Rightarrow k_{t_1} = k_{t_2}$

$$4x = \frac{1}{x} \quad / \cdot x$$

$$4x^2 = 1$$

$$x^2 = \frac{1}{4}$$

$$\boxed{x = \frac{1}{2}}$$

(D)

(11) Derivacija funkcije $f(x) = \operatorname{tg} x + \frac{1}{\cos x}$
(ispit 2 – 3)

$$f'(x) = \frac{1}{\cos^2 x} + (\cos x)^{-1} = \frac{1}{\cos^2 x} - (\cos x)^{-2} \cdot (-\sin x)$$

$$\boxed{f'(x) = \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} = \frac{1 + \sin x}{\cos^2 x}}$$

(A)

DZ

zadaci – 2, 3, 5, 6, 12, 22, 25
ispit 1 – 4, 7, 9
ispit 2 – 7, 9
ispit 3 – 4, 8, 9

24. PRIMJENE DIFERENCIJALNOG RAČUNA

① (Primjer 3)
Odredimo intervale monotonosti i ekstreme funkcije

$$f(x) = \frac{1}{3}x^3 - 2x^2 + 3x$$

Nacrtajmo graf funkcije.

Traženje ekstrema funkcije

$f'(x) = 0$

$f'(x) = x^2 - 4x + 3 = 0 \nearrow x_{1,2} = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2}$ $\begin{matrix} \nearrow x_1 = 3 \\ \searrow x_2 = 1 \end{matrix}\Bigg\}$ stacionarne točke

$f''(x) = 2x - 4$

$f''(x_0 = 3) = 2 \cdot 3 - 4 = 2 > 0$ minimum

$f''(x_0 = 1) = 2 \cdot 1 - 4 = -2 < 0$ maksimum

$T_1(3, 0)$ $T_2\left(1, \frac{4}{3}\right) \rightarrow$ ekstremi

$f(3) = \frac{1}{3} \cdot 3^3 - 2 \cdot 3^2 + 3 \cdot 3 = 0$

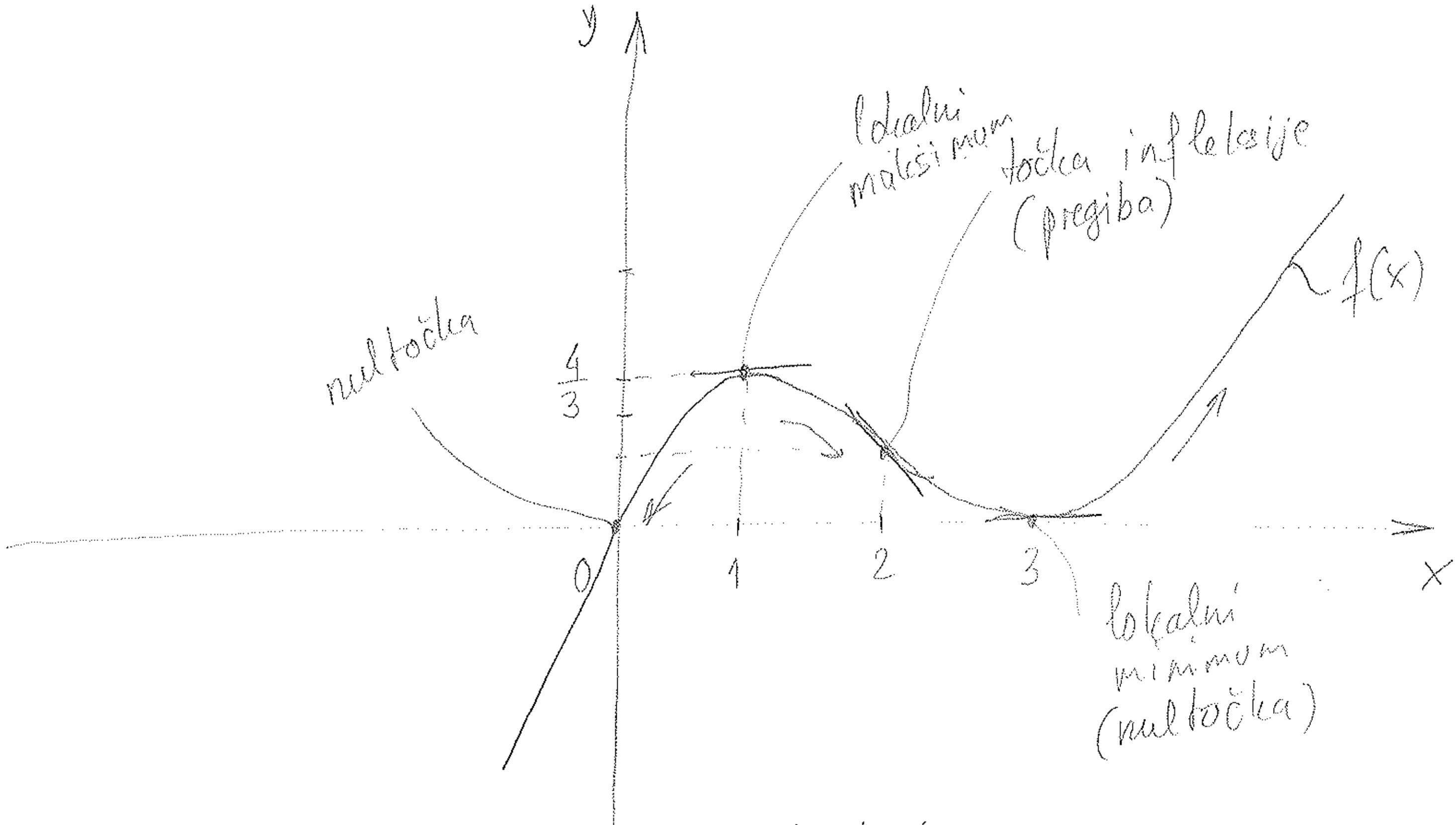
$f(1) = \frac{1}{3} \cdot 1^3 - 2 \cdot 1^2 + 3 \cdot 1 = -\frac{5}{3} + 3 = \frac{4}{3}$

$f''(x) = 0$

$2x - 4 = 0 \Rightarrow x = 2$

$f(2) = \frac{1}{3} \cdot 2^3 - 2 \cdot 2^2 + 3 \cdot 2 = \frac{8}{3} - 8 + 6 = \frac{2}{3}$

$T_3\left(2, \frac{2}{3}\right)$ - točke infleksije



intervali	$f'(x)$	ponašanje funkcije
$\langle -\infty, 1 \rangle$	>0	↗ raste
$\langle 1, 3 \rangle$	<0	↘ pada
$\langle 3, +\infty \rangle$	>0	↗ raste

② (Primjer 4)
Odredimo nultočke, intervale monotonosti i ekstreme funkcije

$$f(x) = (x^3 - x^2)\, e^x$$

Nacrtajmo njen graf.

<u>Nultočke funkcije</u>:

$$f(x) = (x^3 - x^2)\, e^x = x^2 (x-1)\, e^x$$

$x_{1,2} = 0 \qquad y_{1,2} = 0$

$x_3 = 1 \qquad y_3 = 0$

Stacionarne točke:

$$f'(x) = (3x^2-2x)e^x + (x^3-x^2)e^x = (x^3+2x^2-2x)e^x$$

$$f'(x) = 0$$

$$(x^3+2x^2-2x)e^x = 0 \quad / : e^x$$

$$x^3+2x-2x=0$$

$$x(x^2+2x-2)=0$$

$$x_1 = 0$$

$$x_{2,3} = \frac{-2 \pm \sqrt{4+8}}{2} = \frac{-2 \pm \sqrt{12}}{2} = \frac{-2 \pm 2\sqrt{3}}{2}$$

$$x_{2,3} = -1 \pm \sqrt{3}$$

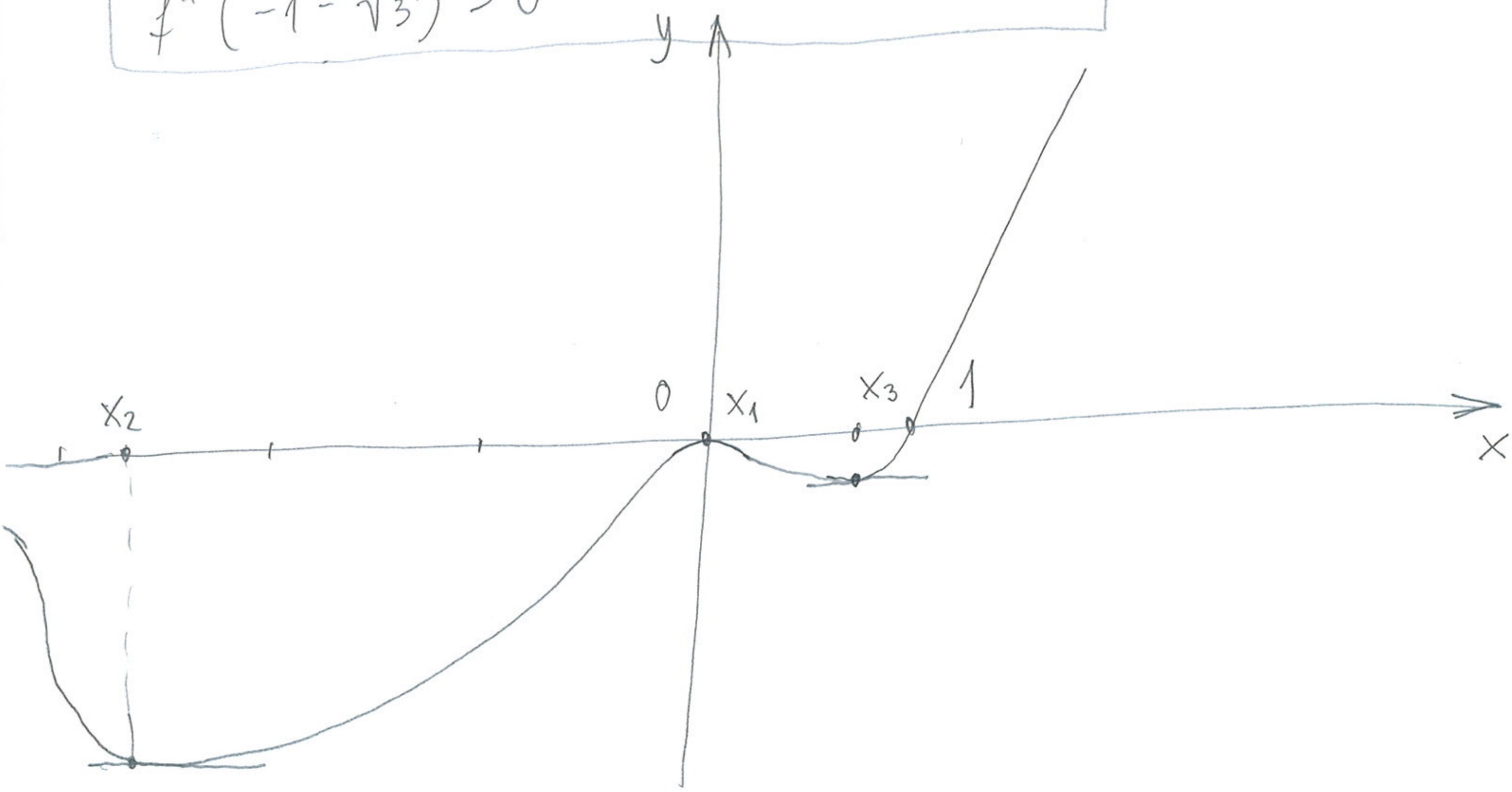
$$\frac{d^2f}{dx^2} = f''(x) = (3x^2+4x-2)e^x + (x^3+2x^2-2x)e^x$$

$$f''(x) = (x^3+5x^2+2x-2)e^x$$

$f''(0) = -2 < 0$ lokalni maksimum

$f''(-1+\sqrt{3}) > 0$ lokalni minimum

$f''(-1-\sqrt{3}) > 0$ -||- -||-



x	f(x)
$-1-\sqrt{3}$	$\approx -427,97$
$-1+\sqrt{3}$	$\approx -0,3$

$\langle -\infty, x_2 \rangle$	$\searrow$	pada
$\langle x_2, 0 \rangle$	$\nearrow$	raste
$\langle 0, x_3 \rangle$	$\searrow$	pada
$\langle x_3, 1 \rangle$	$\nearrow$	raste
$\langle 1, +\infty \rangle$	$\nearrow$	raste

ispitati $f''(x)=0 \rightarrow$ točke infleksije

$x^3+5x^2+2x-2=0 \rightarrow$ riješiti jednadžba!!!

③ (Primjer 6)
U skupu pravokutnih trokuta s hipotenuzom $\sqrt{2}$ cm odredimo trokut s najvećim opsegom.

pitagorin poučak: $a^2+b^2=c^2$

$a^2+b^2=(\sqrt{2})^2=2 \Rightarrow a=\sqrt{2-b^2}$

$O=a+b+c$

$c=\sqrt{2}$

$\sqrt{2-b^2}=(2-b^2)^{\frac{1}{2}}$

$O=\sqrt{2-b^2}+b+\sqrt{2}=f(b)$

$$\frac{dO}{db}=\frac{1}{\cancel{2}}(2-b^2)^{-\frac{1}{2}}\cdot(-\cancel{2}b)+1=\frac{-b}{\sqrt{2-b^2}}+1=0$$

$$\frac{-b}{\sqrt{2-b^2}}+1=0$$

$$\frac{-b}{\sqrt{2-b^2}} = -1 \quad /^2$$

$$\frac{b^2}{2-b^2} = 1$$

$$b^2 = 2 - b^2$$

$$2b^2 = 2 \Rightarrow \underline{b = 1\text{ cm}}$$

provjera:

$$\frac{d^2O}{db^2} = O''(b) = -\frac{2}{2-b}$$

$$\underline{O''(b=1)} = -\frac{2}{2-1} = \underline{-2 < 0} \quad \text{maksimum}$$

$$\underline{a} = \sqrt{2-b^2} = \sqrt{2-1^2} = \underline{1\text{ cm}}$$

<u>Maksimalni opseg</u> je za $\boxed{\begin{array}{l} a = 1\text{ cm} \\ b = 1\text{ cm} \\ c = \sqrt{2}\text{ cm} \end{array}}$

4 (Primjer 7)

Koji od valjaka obujma 1000 cm^3 ima najmanje oplošje?

volumen valjka: $V = r^2\pi v \Rightarrow v = \frac{V}{r^2\pi} = \frac{1000}{r^2\pi}$

oplošje valjka: $P = 2r^2\pi + 2r\pi v$

$$P = 2r^2\pi + 2\cancel{r}\cancel{\pi} \cdot \frac{1000}{r^{\cancel{2}}\cancel{\pi}} = 2r^2\pi + \frac{2000}{r}$$

$$\frac{dP}{dr} = P'(r) = 0$$

$$P'(r) = 4r\pi - \frac{2000}{r^2} = 0$$

$$4r\pi = \frac{2000}{r^2}$$

$$r^3\pi = 500$$

$$r^3 = \frac{500}{\pi} \Big/ ^{\frac{1}{3}}$$

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5{,}42 \text{ cm}$$

provjera:

$$\frac{d^2P}{dr^2} = P''(r) = 4\pi - \left(\frac{2000}{r^2}\right)' = 4\pi - (-2)\cdot 2000\, r^{-3} = 4\pi + \frac{4000}{r^3}$$

$P''(r = 5{,}42 \text{ cm}) > 0$ minimum

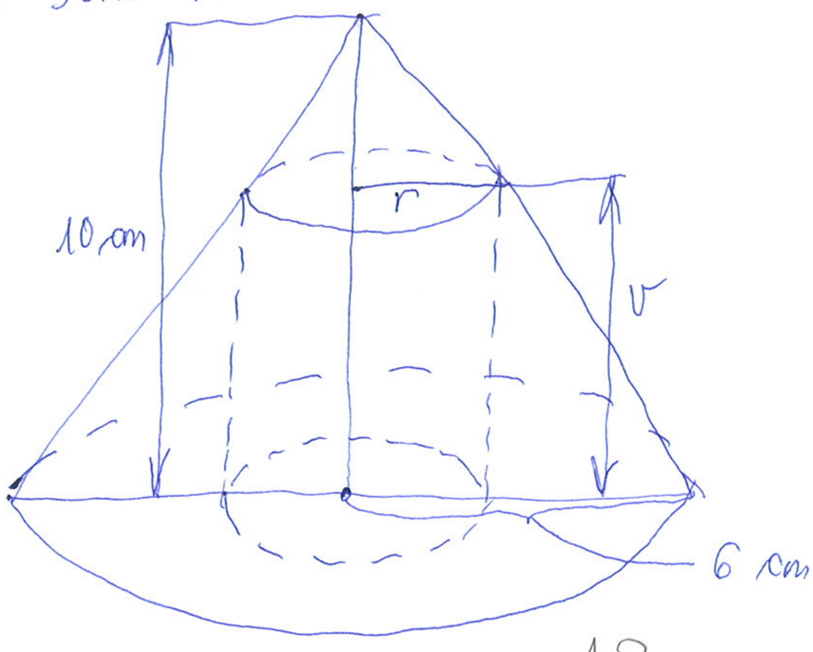
$$\boxed{v = \frac{1000}{r^2\pi} = \frac{1000}{(5{,}42)^2\pi} = 10{,}84 \text{ cm}}$$

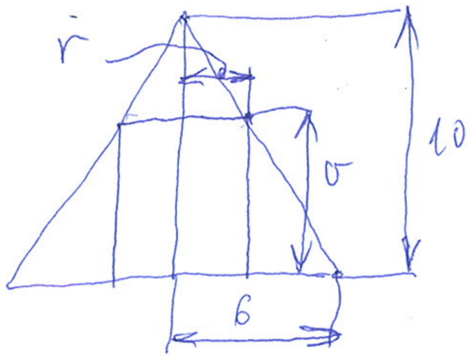
$$\boxed{r = 5{,}42 \text{ cm}}$$

DZ

zadaci – 1, 2, 3, 4, 10, 11, 14, 19, 25, 27, 28, 30
ispit 1 – 1, 3, 8
ispit 2 – 3, 7, 8
ispit 3 – 2, 4, 6, 8

(5) (Pr 8)
Stošcu s polumjerom osnovice 6 cm i visinom 10 cm upisan je valjak kojem baza leži na osnovici stošca. Koji od svih takvih valjaka ima najveći obujam?





$$\frac{10 - v}{r} = \frac{10}{6}$$

$$\Rightarrow v = 10 - \frac{5}{3} r$$

$$V_{valjka} = r^2\pi \cdot v = r^2\pi \cdot \left(10 - \frac{5}{3}r\right) = 10r^2\pi - \frac{5}{3}r^3\pi = f(r)$$

$$\frac{dV_{valjka}}{dr} = 20r\pi - \cancel{3}^{1} \cdot \frac{5}{\cancel{3}_{1}} r^2\pi = 0$$

$$20r\pi = 5r^2\pi \quad / : 5r\pi$$

$$r = 4$$

$$v = 10 - \frac{5}{3}r = 10 - \frac{5}{3} \cdot 4 = 10 - \frac{20}{3} = \frac{10}{3}$$

provjera:

$$\frac{d^2V_{valjka}}{dr^2} = 20\pi - 10r\pi$$

$$\left.\frac{d^2V_{valjka}}{dr^2}\right|_{r=4} = 20\pi - 10 \cdot 4 \cdot \pi = -20\pi < 0$$ radi se o maksimumu

Valjak s polumjerom 4 cm i visinom $v = \frac{10}{3}$ je valjak s najvećom mogućim obujmom kojeg možemo upisati u zadani stožac.