10. Lineami

opecton.

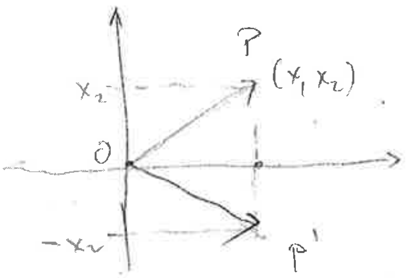
LINEARNI OPERATOR

Neka su X i Y vekt. prostori nad poljem Φ. Tada preslikavanje $A: X \to Y$ nazivamo linearnim operatorom ako vrijedi

$$A(\lambda x + \mu y) = \lambda A x + \mu A y$$

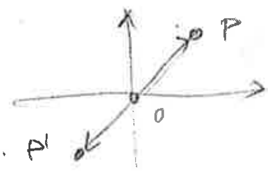
PRIMJERI: $A(x_1 \vec{e}_1 + x_2 \vec{e}_2) = x_1 \vec{e}_1 - x_2 \vec{e}_2$ — osna (simetrija)!

tj. $A: \vec{OP} \to \vec{OP'}$ $\quad f: P \to P' \quad P' = f(P)$



$A(\vec{OP}) = \vec{OP'}$

$f: (x_1, x_2) \to (x_1, -x_2)$



$f: (x_1, x_2) \mapsto (-x_1, -x_2)$

- Centralna simetrija

pripada joj operator $A(x_1 \vec{e}_1 + x_2 \vec{e}_2) = -x_1 \vec{e}_1 - x_2 \vec{e}_2$

- HOMOTETIJA $(x_1, x_2) \to (\lambda x_1, \lambda x_2)$ $\quad A(x_1 \vec{e}_1 + x_2 \vec{e}_2) = \lambda x_1 \vec{e}_1 + \lambda x_2 \vec{e}_2$

MATRICE — zapis lin. operatora u bazi!

Neka je u $D_2 = \{\vec{OP} : P \in V_2\}$ zadana baza $(\vec{e}_1, \vec{e}_2)$. Tada je linearni operator $A: D_2 \to D_2$ potpuno određen sa $A\vec{e}_1$ i $A\vec{e}_2$ kao slikama operatora na baznim vektorima!

$$Ae_1 = a_{11} \vec{e}_1 + a_{21} \vec{e}_2 \qquad A = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$Ae_2 = a_{12} \vec{e}_1 + a_{22} \vec{e}_2$$

Shemu zovemo matrica operatora A u bazi $(\vec{e}_1, \vec{e}_2)$ pridruženu operatoru A.

PRIMJER: za $A(x_1, x_2) = (x_1, -x_2)$ nađimo pripadnu matricu.

$$A(x_1 \vec{e}_1 + x_2 \vec{e}_2) = x_1 \vec{e}_1 - x_2 \vec{e}_2 \Rightarrow A\vec{e}_1 = e_1, \quad A\vec{e}_2 = -e_2 \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

PRIMJERI Zadanoj matrici $A=\begin{bmatrix}3 & -1\\ 1 & 2\end{bmatrix}$ u bazi $(\vec{e}_1, \vec{e}_2)$ prostora D_2 pridružite lin. operator $A: D_2 \to D_2$

$$Ae_1 = 3e_1 + e_2 \qquad A(x_1\vec{e}_1 + x_2\vec{e}_2) = x_1 Ae_1 + x_2 Ae_2 =$$
$$Ae_2 = -e_1 + 2e_2 \qquad = x_1(3e_1 + e_2) + x_2(-\vec{e}_1 + 2e_2)$$
$$= (3x_1 - x_2)\vec{e}_1 + (x_1 + 2x_2)\vec{e}_2$$

$$A(x_1, x_2) = (3x_1 - x_2,\ x_1 + 2x_2)$$

Zbrajanje: matrica ima svojstva — Jednakost matrica $\{a_{ij}\} = \{b_{ij}\}$ $a_{ij} = b_{ij}$

1) $(A+B)+C = (A+B)+C$

2) $A + 0 = A$

3) $A + (-A) = 0$

4) $A + B = B + A$

$$\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix} + \begin{bmatrix}1 & 2\\ -1 & 3\end{bmatrix} = \begin{bmatrix}2 & 1\\ 1 & 6\end{bmatrix}$$

$$\{c_{ij}\} = \{a_{ij}\} + \{b_{ij}\} = \{a_{ij} + b_{ij}\}$$

Množenje matrica sa skalarom: $\lambda A = \lambda\{a_{ij}\} = \{\lambda a_{ij}\}$

5) $\lambda(A+B) = \lambda A + \lambda B$

6) $(\lambda + \mu)A = \lambda A + \mu A$

7) $\lambda(\mu A) = (\lambda\mu)A$

8) $1 \cdot A = A$

Množenje matrica: $A \cdot B = C \qquad C = \{c_{ij}\}$

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

PRIMJER: $A = \begin{bmatrix}1 & -2\\ 2 & -3\end{bmatrix} \qquad B = \begin{bmatrix}-1 & 2\\ -3 & 3\end{bmatrix}$

$$AB = \begin{bmatrix}1 & -2\\ 2 & -3\end{bmatrix}\begin{bmatrix}-1 & 2\\ -3 & 3\end{bmatrix} = \begin{bmatrix}5 & -4\\ 7 & -5\end{bmatrix} \qquad BA = \begin{bmatrix}-1 & 2\\ -3 & 3\end{bmatrix}\begin{bmatrix}1 & -2\\ 2 & -3\end{bmatrix} = \begin{bmatrix}3 & -4\\ 3 & -3\end{bmatrix}$$

Vidimo da $AB \neq BA$

$A = \begin{bmatrix}1 & 2\\ -2 & 0\end{bmatrix} \qquad AB = BA = \begin{bmatrix}-7 & -6\\ 6 & -4\end{bmatrix}$

$B = \begin{bmatrix}-3 & 2 \\ \ldots\end{bmatrix}$ Komutativne matrice

PRIMJER: $\begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $AB = 0$!

Produkt matrica ima svojstvo:

1) $(AB)C = A(BC)$

2) $C(A+B) = CA + CB$

3) $(A+B)D = AD + BD$

PRIMJER: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Potencije matrice.

$A^{n+1} = A \cdot A^n$, $A^n \cdot A^m = A^{n+m}$ $(A^p)^q = A^{p \cdot q}$

Kurepa (Str 127)

DETERMINANTE: INVERZNA MATRICA

DEF! Za matricu A kažemo da je regularna matrica ako postoji matrica B tako da je

$$AB = BA = I \qquad (x \cdot x^{-1} = 1 \;!)$$

tada matricu B zovemo inverzna matrica A^{-1}

tj. $\boxed{A \cdot A^{-1} = A^{-1} A = I}$

Svojstva: $(A^{-1})^{-1} = A$

$(AB)^{-1} = B^{-1} A^{-1}$ pokažimo: $B^{-1} A^{-1} (AB) = B^{-1} (A^{-1} A) B = I$

$A^{-n} = (A^{-1})^n$

Potražimo A^{-1} za $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $A^{-1} = \begin{bmatrix} x & u \\ y & v \end{bmatrix}$

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x & u \\ y & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{matrix} a_{11} x + a_{12} y = 1 \\ a_{21} x + a_{22} y = 0 \end{matrix}$ $\begin{matrix} a_{11} u + a_{12} v = 0 \\ a_{21} u + a_{22} v = 1 \end{matrix}$

$(a_{11} a_{22} - a_{21} a_{12})\, x = a_{22}$ $(a_{11} a_{22} - a_{21} a_{12})\, u = -a_{12}$

$(a_{11} a_{22} - a_{21} a_{12})\, y = \ldots$

Stavimo $D = \det A = a_{11} a_{22} - a_{21} a_{12} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ — znamo determinanta kvadratne matrice 2-reda

te je $A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

Za postojanje A^{-1} mora biti $\det A \neq 0$ (regularnost A)

Svojstva determinanti:

$\det(AB) = \det A \cdot \det B$

Razvoj det: $D = \sum a_{ij} A_{ij}$

$A_{ij} = (-1)^{i+j} D_{ij}$ – algebarski komplement KOFAKTOR

1) Elementi redka i stupca svi jednaki nuli $D = 0$

2) Ako dva susjedna redka ili stupca zamijene mjesto det. promijeni predznak (ili proporcionalni)

3.) Ako imamo dva jednaka stupca ili redka $D = 0$.

4) Ako nekom stupcu ili redku dodamo linear. kombi. ostalih redaka ili stupaca det. ne mijenja vrijednost

5. Ako je neki stupac ili redak jednak linea. kombi. ostalih stup. ili redaka $D = 0$

6. $\det A = \det A^T$

7. Zajednički faktor svih elemenata redka ili stupca može se izlučiti ispred det.

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \frac{A^*}{\det A}$$

RAZVOJ det:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

Linearni sistemi

$$x_1 + 2x_2 + 3x_3 = 0$$
$$2x_1 + 4x_2 + 5x_3 = 1$$
$$3x_1 + 5x_2 + 6x_3 = 2$$

$$Ax = b$$
$$x = A^{-1}b$$

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix} \quad - - - \quad A^{-1}b = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}$$

ZADAĆU:
$$x_1 - 2x_2 + x_3 = 0$$
$$2x_1 + x_2 + 3x_3 = 13$$
$$-x_1 + 2x_2 + x_3 = 6$$

CRAMEROVO PRAVILO

$$Ax = b \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad x_i = \frac{\det A_i}{\det A} = \frac{D_i}{D}$$

I) $D \neq 0$ sistem je određen (ima rješenje)

II) $D = 0$ i $D_i \neq 0$ za neko $i = 1, 2, 3$ nemoguć

III) $D = 0$ i $D_1 = D_2 = D_3 = 0$ sistem je neodređen ili je sistem nemoguć

HOMOGEN SISTEM

$Ax = 0$ sistem ima rješenje $\neq 0$ za $\det A = 0$

Znamo da jednoznačnost znači da lin. op $A: 0 \to 0$ samo onda ako je on regularan tj. A^{-1} postoji tj. za $\det A \neq 0$ samo trivijalno rješenje.

Za sing. operator mijenja da za $x \neq 0$ $Ax = 0$

GAUSS-ova metoda eliminacije

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ 2x_1 + 5x_2 \quad &= 1 \\ x_1 + 2x_2 + x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & 5 & 0 & 1 \\ 1 & 2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -1 & 2 & -3 \\ 0 & 1 & -2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & -7 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + 5x_3 &= -7 \\ x_2 - 2x_3 &= 3 \end{aligned}$$

Stavimo $x_3 = \lambda \in \mathbb{R}$

$$x_1 = -5\lambda - 7$$
$$x_2 = 2\lambda + 3$$
$$x_3 = \lambda$$

$$x = (-5\lambda - 4, 2\lambda + 3, \lambda)$$

$$x = \lambda(-5, 2, 1) + (-4, 3, 0)$$

HOMOGEN SISTEM: $Ax = 0$

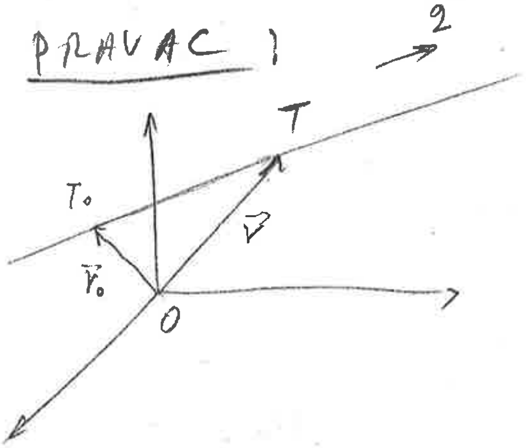
$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 5 & 0 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -5\lambda$$
$$x_2 = 2\lambda$$
$$x_3 = \lambda$$

$$x = \lambda(-5, 2, 1)$$

Kontrola: $Ax = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 5 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

PRAVAC:



$$\vec{OT} = \vec{OT_0} + \lambda \vec{a}$$

$T(x, y, z)$ $\quad T_0(x_0, y_0, z_0)$ $\quad \vec{a} = (p, q, r)$

$$x - x_0 = \lambda p$$
$$y - y_0 = \lambda q$$
$$z - z_0 = \lambda r$$

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r} = \lambda$$

PRIMJER: Pravac kroz dvije točke $\vec{q} = \vec{T_1T_2}$

$T_1(x_1, y_1, z_1)$ $\quad T_2(x_2, y_2, z_2)$

PRIMJER Odredite sjecište pravca p koji prolazi točkama $T_1(2, 5, -3)$ i $T_2(3, 4, 1)$ sa ravninom $3x + 2y - z = 10$

$\boxed{S(5, 2, 9)}$

$$\frac{x-2}{1} = \frac{y-5}{-1} = \frac{z+3}{4} = \lambda$$

$x = 2 + \lambda$
$y = 5 - \lambda$
$z = -3 + 4\lambda$

$$3(2+\lambda) + 2(5-\lambda) - (-3+4\lambda) = 10 \implies \lambda = 3$$

PRIMJER p: pravac sjecište ravnina

$$2x - 3y - 3z - 9 = 0$$
$$x - 2y + z + 3 = 0$$

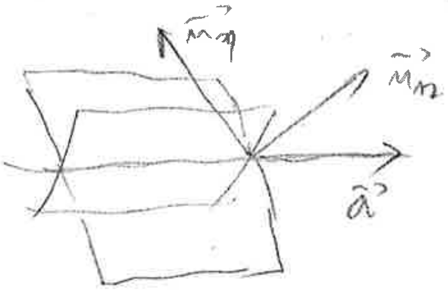
$$2x - 3y = 3z + 9$$
$$x - 2y = -z - 3 \quad /(-2)$$

$y = 5z + 15$
$x = 9z + 27$

Stavimo $z = t \in R$

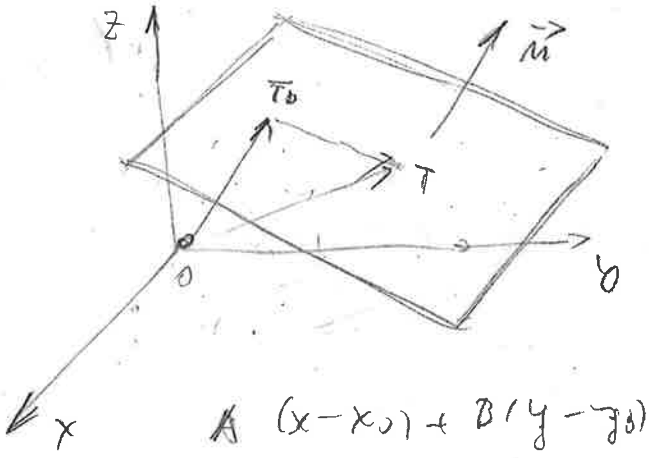
$$\vec{a} = (9, 5, 1)$$

$$\frac{x - 27}{9} = \frac{y - 15}{5} = \frac{z - 0}{1}$$



$$\vec{a} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & -3 & -3 \end{vmatrix} = (9, 5, 1)$$

ANALITIČKA GEOM. PROSTORA



$$\vec{r_0} = \overrightarrow{OT_0} = (x_0, y_0, z_0)$$

$$\vec{r} = \overrightarrow{OT} = (x, y, z)$$

$$\vec{n} = (A, B, C)$$

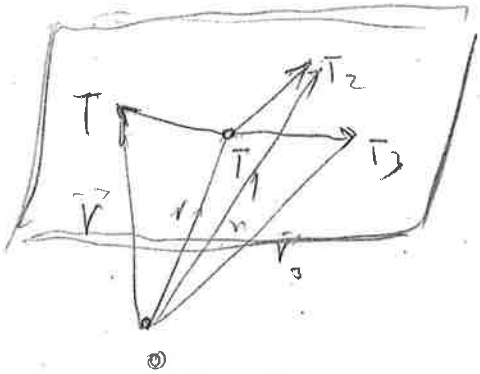
$$\boxed{\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0}$$

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

$$\vec{n}\vec{r} - \vec{n}\vec{r_0} = 0 \quad \Rightarrow \quad \boxed{\vec{n}\vec{r} + D = 0}$$

$$Ax + By + Cz + D = 0$$

PRIMJER: Odredite jednadžbu ravnine koja prolazi kroz točke $T_1(1,-1,2)$, $T_2(3,0,1)$, $T_3(0,0,-2)$



Uvjet da tri vektora leže u ravnini je $\overrightarrow{T_1T} \cdot (\overrightarrow{T_1T_2} \times \overrightarrow{T_1T_3}) = 0$ $\quad T = (x, y, z)$

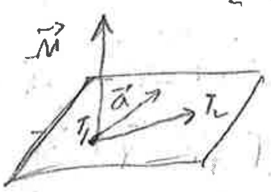
$$\begin{vmatrix} x-1 & y+1 & z-2 \\ 2 & 1 & -1 \\ -1 & 1 & -4 \end{vmatrix} = 0$$

$$-4(x-1) + y + 1 + 2(z-2) + z - 2 + x - 1 + 8(y+1) = 0$$

$$-4x + 4 + y + 1 + 2z - 4 + z - 2 + x - 1 + 8y + 8 = 0$$

$$-3x + 9y + 3z + 6 = 0 \qquad \boxed{x - 3y - z - 2 = 0}$$

PRIMJER: Odredi jed. ravnine kroz točke $T_1(1,-1,2)$ $T_2(0,-1,1)$ i $\parallel$ sa $\vec{a} = (1, 1, -1)$



$$\vec{n} = \overrightarrow{T_1T_2} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = (1, -2, -1)$$

$$x - 2(y+1) - (z-1) = 0$$

$$x - 2y - z - 1 = 0$$

kontrola $\vec{a} \perp \vec{n}$ $\quad \vec{a} \cdot \vec{n} = 0$

(7) Dokaži

$$D = \begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a)$$

$$-8abc + (a+b)(b+c)(c+a) + (a+c)(b+a)(c+b) + 2b(a+c)^2 + 2a(b+c)^2 + 2c(a+b)^2$$

$$D = -8abc + 2(a+b)(b+c)(c+a) + 2(a^2b + 2abc + bc^2 + ab^2 + 2abc + ac^2 +$$

$$+ a^2c + 2abc + b^2c)$$

$$= 2[(abc + bc^2 + ac^2 + b^2c) + (abc + a^2b + ab^2 + a^2c)] + 2(a+b)(b+c)(c+...$$

$$2[(ab + bc + ac + b^2)c + a(bc + ab + b^2 + ac)] =$$ samo uglata zagrada

$$= 2(ab + ac + bc + b^2)(a+c) = 2[a(b+c) + b(b+c)](a+c)$$

$$= 2(a+b)(b+c)(c+a) \quad \Rightarrow \quad D = 4(a+b)(b+c)(c+a)$$

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = abc$$ trokutna forma