Eksponencijalne

i

[illegible]

funkcije

4. Eksponencijalne funkcije

Ekspanencijalna funkcija

o egzistenciji ex. funkcije

Teorem: $f: R \to R \quad a>0 \text{ i } a \neq 1$

1) $f(x) > 0 \quad \forall x \in R$

2) $f(x+y) = f(x) f(y) \quad \forall x, y \in R$

3) za $a>1$ f strogo raste

$a<1$ f strogo pada

4) $\forall y_0 > 0 \quad \exists!\, x_0 : f(x_0) = y_0$

tj $f: R \to R^+$ (strogo pozitivni)

5) $\forall m \in Z \quad f(m) = a^m$

Definicija: opće eksponencijalne funkcije baze a oznaka a^x ili $f(x) = a^x$

1) $a^x > 0 \quad \forall x \in R$

2) $a^{x+y} = a^x \cdot a^y \quad \forall x, y \in R$

3) $x<y \Rightarrow a^x < a^y$ ako je $a>1$

$x<y \Rightarrow a^x > a^y$ ako je $a<1$.

4) $\forall y_0 > 0 \quad \exists!\, x_0 \in R : a^{x_0} = y_0$

Primjedba: $(a^x)^y = a^{xy} \quad \forall x, y \in R$

Definicija inverzne funkcije opće eksponencijalne funkcije $f: x \mapsto a^x$ zove se logaritamska funkcija baze a i označava se $f^{-1}: x \mapsto \log_a x$

Teorem: $f^{-1}: x \mapsto \log_a x \qquad f^{-1}: R^+ \to R$

1) $\log_a (xy) = \log_a x + \log_a y \quad \forall x, y > 0 \quad x, y \in R^+$

2) $\log_a 1 = 0$

3) $\log_a$ je strogo rastuća za $a>1$, odnosno strogo padajuća za $a<1$

svojstva: $y = f(x) = a^x \Rightarrow x = f^{-1}(y) = \log_a y \Rightarrow \begin{cases} x = \log_a (a^x) & \forall x \in R \\ y = a^{\log_a y} & \forall y > 0 \end{cases}$

$x \mapsto a^x \qquad \overset{a=e}{x \mapsto e^x} \qquad \overset{a=10}{x \mapsto 10^x}$

$x \mapsto \log_a x \qquad x \mapsto \ln x \qquad x \mapsto \log x$

Veza opće ex. funkcije $x \mapsto a^x$ i ex. funkcije $x \mapsto e^x$ je

zbog $y = a^{\log_a y}$ stavimo $a = e \Rightarrow y = e^{\ln y}$ i $y = a \Rightarrow \boxed{a = e^{\ln a}}$

kako je $(a^x)^y = a^{xy} \Rightarrow a^x = e^{x \ln a} = \exp(x \ln a) \quad \forall x \in R$

Opća potencija: zbog $(a^x)^y = a^{xy}$ i $\forall t > 0 \Rightarrow t = e^{\ln t}$ može se definirati $x \mapsto x^c \quad \forall c \in R$ i $\forall x > 0$ prema $x^c = e^{c \ln x} \quad \forall x > 0$

svojstvo: $(xy)^c = e^{c \ln(xy)} = e^{c\ln x + c \ln y} = e^{c \ln x} \cdot e^{c \ln y} = x^c \cdot x^y$

eksponencijalna funkcija $f: x \mapsto e^x$ ↯ $y = e^x \quad f: R \to R^+$

Veza između logaritamske funkcije $\log_a x$ i funkcije $\ln x$

zbog $x = a^{\log_a x}$ i $x = e^{\ln x} \Rightarrow$ zbog $a = e^{\ln a}$ potenciranjem s $\log_a x$

$x = a^{\log_a x} = e^{\log_a x \cdot \ln a} = e^{\ln x} \Rightarrow \boxed{\ln x = \ln a \cdot \log_a x}$

za $a = 10 \Rightarrow \ln x = \frac{1}{M} \log x \qquad M = \frac{1}{\ln 10} = \log e = 0{,}4343$

$\frac{1}{M} = \ln 10 = \frac{1}{\log e} \approx 2{,}3 \qquad \boxed{M = \log e \approx 0{,}43} \qquad \boxed{\log x = M \ln x}$

Veza $\log_a$ i $\log_b$

po definiciji je: $b = a^{\log_a b} \Rightarrow x = a^{\log_a x} = b^{\log_b x} = a^{\log_b x \cdot \log_a b}$

zbog $(a^x)^y = a^{xy}$

i odatle $\boxed{\log_a x = \log_a b \cdot \log_b x}$

specijalno za $x = a \Rightarrow \log_a b \cdot \log_b a = 1$

$a = a^{\log_a a} \Rightarrow \log_a a = 1$

ZADACI

① Veza $\log_a$ i $\ln$ (II) a^x i e^x (III) $\log_a$ i $\log_b$

zbog $a = e^{\ln a} \Rightarrow a^x = e^{x\ln a}$ (II) $x = e^{\ln x} \Rightarrow x^c = e^{c\ln x}$ $\forall x>0$

(III) $x = a^{\log_a x} \Rightarrow \log_b x = \log_a x \cdot \log_b a \Rightarrow \log_a x = \frac{1}{\log_b a}\log_b x$

(I) $x = a^{\log_a x} \Rightarrow \ln x = \log_a x \cdot \ln a \Rightarrow \log_a x = \frac{1}{\ln a}\ln x$ $\forall x>0$

$\frac{1}{\ln a} = M$ modul logaritma baze a za $a=10$ $\log x = \frac{1}{\ln 10}\ln x$

$M = 1/\ln 10 = 2,3$ tj $\ln x = \frac{1}{M}\log x$ $\log x = M\ln x$

$y = \log_a x \Rightarrow x = a^y$
$\ln x = y\ln a$
$y = \frac{1}{\ln a}\ln x = \log_a x$

veza prirodnih i dekadskih logaritama.

stavimo $\log_b a = M \Rightarrow \log_b x = M\log_a x$ tj. $\log_a x = \frac{1}{M}\log_b x$

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② Ako je $(a^x)^y = a^{xy}$ pokaži da je $\log a^x = x\log a$

$y = \log a^x \Rightarrow a^x = 10^y$ $a = 10^{\log a} \Rightarrow a^x = 10^{x\log a}$

zbog $10^y = 10^{x\log a} \Rightarrow y = x\log a$

ili direktno zbog $a = 10^{\log a} \Rightarrow \left\{ \begin{array}{l} a^x = 10^{\log a^x} \\ a^x = 10^{x\log a} \end{array} \right\} \Rightarrow \log a^x = x\log a$

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③ Neka je $f(x) = (a^n - x^n)^{1/n}$ $0\le x\le a$ $n\in N$

Dokaži da je $(f\circ f)(x) = x$

f je definirana na segmentu $[0,a]$ tj $D(f) = [0,a]$

Kako je $0 \le (a^n-x^n)^{1/n} \le a$ jer je $0\le a^n - x^n \le a^n \Rightarrow 0\le x^n \le a^n \Rightarrow$

$0\le x\le a$ tj i $R(f) = [0,a]$ Sada je $(f\circ f)(x) = f(f(x)) =$

$= \{a^n - [f(x)]^n\}^{1/n} = \{a^n - (a^n - x^n)\}^{1/n} = x \Rightarrow (f\circ f)(x) = x$

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④ Ako je $y_1 = \log_a x$ $y_2 = \log_{a^n} x$ dokaži da je $y_1 = n y_2$

R: $x = a^{y_1}$ i $x = a^{y_2 n} \Rightarrow y_1 = n y_2$

Specijalan slučaj dokaži $\log_{a^n} x = \log_a x^{\frac{1}{n}}$ stavimo $y = \log_{a^n} x \Rightarrow$

$x = a^{ny} \Rightarrow a^y = x^{\frac{1}{n}} \Rightarrow y = \log_a x^{\frac{1}{n}} \Rightarrow \log_{a^n} x = \log_a x^{\frac{1}{n}}$

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⑤ Za $f(x) = \ln x$ naći! $\frac{f(x+t)-f(x)}{t}$

$\frac{\ln(x+t)-\ln x}{t} = \frac{1}{t}\ln\frac{x+t}{x} \Rightarrow \ln\left(1+\frac{t}{x}\right)^{\frac{1}{t}} = \ln\left[\left(1+\frac{1}{\frac{x}{t}}\right)^{\frac{x}{t}}\right]^{\frac{1}{x}}$

zbog $e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ za $t\to 0 \Rightarrow \frac{x}{t}\to\infty$ $x\in R$ pa

$\left(1+\frac{1}{\frac{x}{t}}\right)^{\frac{x}{t}} \to e$ za $t\to 0$ i $\ln e^{\frac{1}{x}} = \frac{1}{x}$ time smo pokazali da je zbog $\lim_{t\to 0}\frac{f(x+t)-f(x)}{t} = f'(x)$

da je derivacija od $\ln x$ tj $(\ln x)' = \frac{1}{x}$ ovo se kasnije dokazuje teoremom o ex. funkciji i njezinoj derivaciji

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$y = \log_a x = \log_{a^n} x^n$ Pokaži! $x = a^y \Rightarrow x^n = a^{ny} \Rightarrow y = \log_{a^n} x^n$

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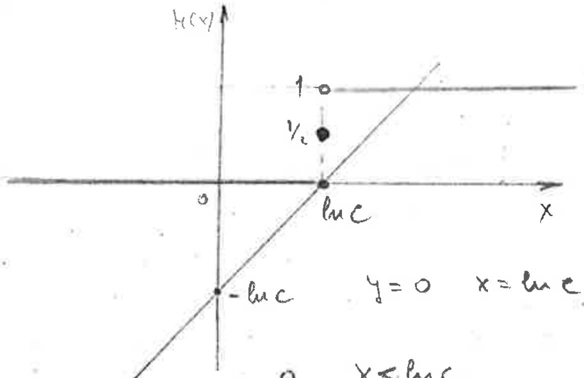
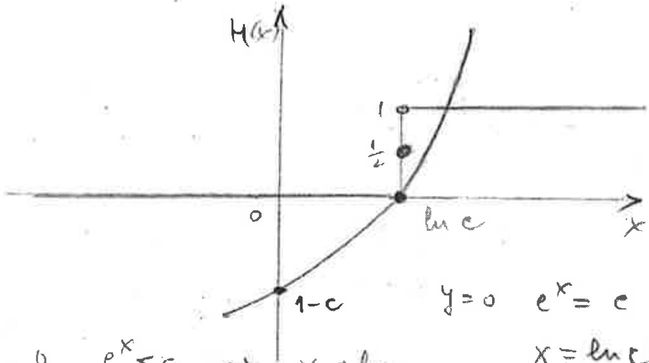
⑨ Dokaži identitet $H(x-\ln c) = H(e^{x}-c)$ $(c>0)$

ako je $H(x) = \begin{cases} 0 & x<0 \\ \frac{1}{2} & x=0 \\ 1 & x>0 \end{cases}$ tj. tzv. Heaviside-ova funkcija

Nacrtajmo $y = x - \ln c$ Nacrtajmo $y = e^{x} - c$



$y=0$ $x=\ln c$



$y=0$ $e^{x}=c \Rightarrow$
$x=\ln c$

$H(y) = \begin{cases} 0 & x<\ln c \\ \frac{1}{2} & x=\ln c \\ 1 & x>\ln c \end{cases}$ $H(y) = \begin{cases} 0 & e^{x}<c \Rightarrow x<\ln c \\ \frac{1}{2} & e^{x}=c \Rightarrow x=\ln c \\ 1 & e^{x}>c \Rightarrow x>\ln c \end{cases}$

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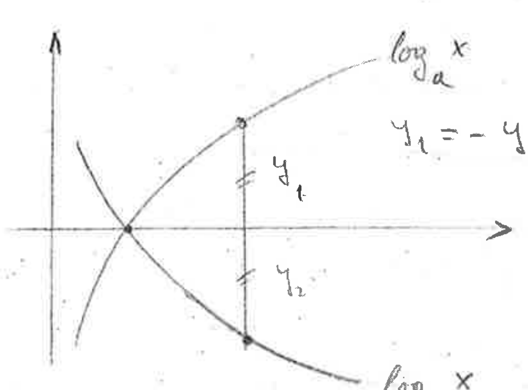
⑩ Dokaži $\log_a x + \log_{\frac{1}{a}} x = 0$ i daj grafičku interpretaciju

(I) Stavimo $y_1 = \log_a x$ i $y_2 = \log_{\frac{1}{a}} x \Rightarrow x = a^{y_1}$ i $x = \left(\frac{1}{a}\right)^{y_2} \Rightarrow x = a^{-y_2}$

tj. $y_1 = -y_2 \Rightarrow y_1 + y_2 = 0$



(II) Zbog $\log_a x \cdot \log_x a = 1 \Rightarrow$

$\log_{\frac{1}{a}} x = \dfrac{1}{\log_x \frac{1}{a}} = -\dfrac{1}{\log_x a} = -\log_a x$

tj. $\log_{\frac{1}{a}} x + \log_a x = 0$ što je i trebalo pokazati.

(III) iz $x = b^{\log_b x} \Rightarrow \log_b x \log_a b = \log_a x$

stavimo $b = \frac{1}{a} \Rightarrow \log_a x = \log_{\frac{1}{a}} x \cdot \log_a \frac{1}{a} = \log_{\frac{1}{a}} x\,(0 - \log_a a)$

$\Rightarrow \log_a x = -\log_{\frac{1}{a}} x \Rightarrow \log_a x + \log_{\frac{1}{a}} x = 0$ kao i prije.

(IV) $\log_a 1 = 0 = \log_a \frac{x}{x} = \log_a x \cdot x^{-1} = \log_a x + \log_a x^{-1} = \log_a x + \log_{\frac{1}{a}} x = 0$

Zbog $\log_a x = \log_{a^m} x^m$ (za $m=-1$)

Primjedba: $\log_{\frac{1}{a}} x = -\log_a x$ npr. za $a = 0{,}1$ imamo $a = \frac{1}{10}$

① $\log_{0{,}1} x = -\log x$ ② $\log_{\frac{1}{3}} x = -\log_3 x$

Dokaži: $\log_b a = \dfrac{\log a}{\log b}$ iz $a = b^{\log_b a} \Rightarrow$ logaritmiranjem

$\log a = \log_b a \cdot \log b$

⑥ Izračunaj explicitno: $\log(x-1)+\log(y+1)=1$

$\log[(x-1)(y+1)] = \log 10 \Rightarrow (x-1)(y+1)=10 \quad y=\frac{10}{x-1}-1 = \frac{11-x}{x-1}$

(A) Riješi: $\log_x 3 = \log_y 9 \qquad x^2-7y^2+26=0$

$\log_{x^2} 9 = \log_y 9 \Rightarrow y=x^2$ i $y-7y^2+26=0 \qquad 7y^2-y-26=0$

$y=\frac{1\pm\sqrt{1+28\cdot 26}}{14} = \frac{1\pm\sqrt{729}}{14} = \frac{1\pm 27}{14} \qquad y=2 \quad x=\sqrt{2}$

(B) Riješi: $x^2 = 1+6\log_4 y$ (1)

$y^2 = 2^x y + 2^{2x+1}$ (2) $\Rightarrow y^2 - y\,2^x - 2^{2x+1}=0$

$y=\frac{1}{2}\left(2^x \pm \sqrt{2^{2x}+2^{2x+3}}\right) = \frac{2^x}{2}\left(1\pm\sqrt{1+2^3}\right) = \frac{2^x}{2}(1\pm 3)$

$y_1 = 2^{x_1+1} \qquad \cancel{y_2 = -2^x}\ (y>0) \quad y=4^{\frac{x+1}{2}} \Rightarrow x^2 = 1+6\,\frac{x+1}{2} = 1+3x+3$

$x^2-3x-4=0 \Rightarrow x_1=-1 \quad x_2=4$

$y_1 = 1 \quad y_2 = 2^5$ zadovoljava zadanu jednadž.

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⑦ Dali su parne ili neparne funkcije $f(x)=\ln\frac{1+x}{1-x} \qquad f(x)=\ln(x+\sqrt{1+x^2})$

(1) $f(-x)=\ln\frac{1-x}{1+x} = -\ln\frac{1+x}{1-x} = -f(x)$ neparna! skiciraj graf!

(2) $f(-x)=\ln(-x+\sqrt{1+(-x)^2}) = \ln(-x+\sqrt{1+x^2}) = \ln\frac{1}{x+\sqrt{1+x^2}}$

$f(-x) = -\ln(x+\sqrt{1+x^2}) = -f(x)$ neparna

$y=f(x) \Rightarrow x=f^{-1}(y)$ tj. $y=f^{-1}(x)$

⑧ Naći inverznu funkciju za funkcije:

(I) $y=1+\log(x+1)$, (II) $y=\log\frac{x}{2}$, (III) $y=\frac{10^x-10^{-x}}{10^x+10^{-x}}$ (IV) $y=\sqrt{1-\log_2(x-1)}$

(I) $\log(x+1) = y-1 \Rightarrow x+1=10^{y-1} \quad x=10^{y-1}-1$ ili $y=10^{x-1}-1$

(II) $y=\log\frac{x}{2} \Rightarrow \frac{x}{2}=10^y \quad x=2\cdot 10^y$ ili $y=2\cdot 10^x$

(III) $y=f(x) \Rightarrow$ inverzna $x\cdots y$ i $y\cdots x$ tj. $x=f(y)$ je inverzna $y=f?$

isto što i $y=f(x) \Rightarrow x=f^{-1}(y)$ ili $y=f^{-1}(x)$ tj. inverzna je:

$x=\frac{10^y-10^{-y}}{10^y+10^{-y}} \Rightarrow \frac{10^{2y}-1}{10^{2y}+1}=x \qquad 1-\frac{10^{2y}-1}{10^{2y}+1} = 1-x$

$\frac{2}{10^{2y}+1} = 1-x \Rightarrow 10^{2y}+1 = \frac{2}{1-x} \qquad 10^{2y} = \frac{2}{1-x}-1 = \frac{1+x}{1-x}$

$y=\frac{1}{2}\log\frac{1+x}{1-x}$ inverzna funkcija.

(IV) za $y=\sqrt{1-\log_2(x-1)}$ inverzna je iz $x=\sqrt{1-\log_2(y-1)}$

$\Rightarrow x^2 = 1-\log_2(y-1) \qquad \log_2(y-1) = 1-x^2 \qquad y-1 = 2^{1-x^2} \qquad y=1+2^{1-x^2}$

(V) $y=\frac{1}{\log(x-2)}$ inverzna iz $x=\frac{1}{\log(y-2)} \qquad \log(y-2)=\frac{1}{x}$

$y-2 = 10^{\frac{1}{x}} \qquad y = 2+10^{\frac{1}{x}}$ inverzna funkcija.

Odredi područje definicije funkcije $y=\sqrt{\log\frac{5x-x^2}{4}}$ $(1\le x\le 4)$

$$y = e^{-\sqrt{\frac{x-1}{x+1}}} \qquad \frac{x-1}{x+1} = 1 + \frac{-2}{x+1} \approx 1 + \frac{-2}{x}$$

$x \to \infty \quad y \to \frac{1}{e}+ \quad \varepsilon > 0$

$e^{-(1-\varepsilon)} = e^{-1} e^{\varepsilon} > \frac{1}{e}$

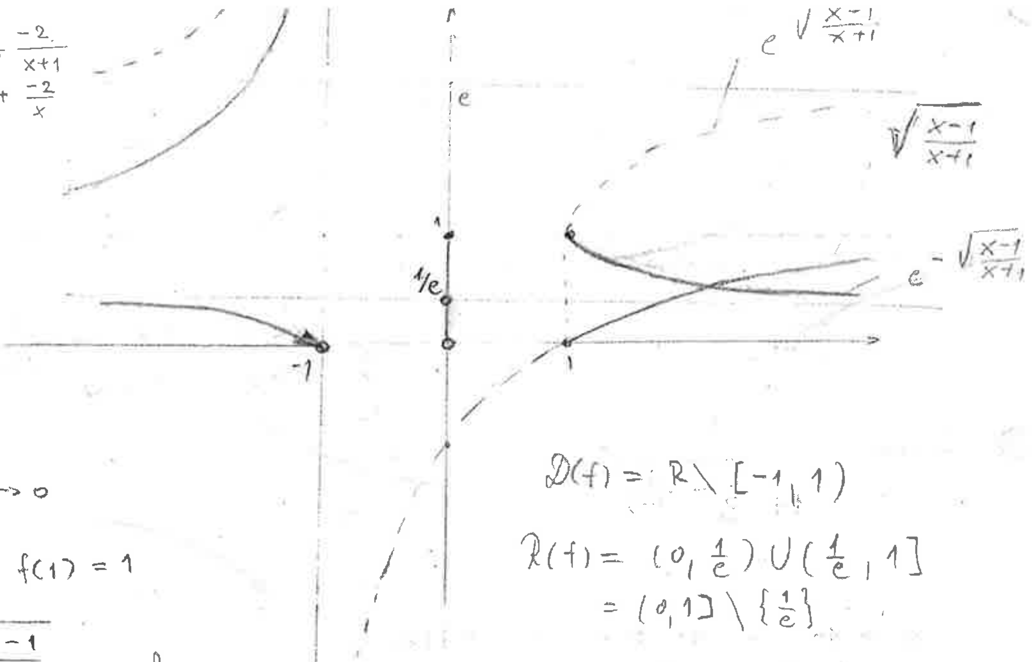
$x \to -\infty \quad y \to \frac{1}{e}-$ zbog

$e^{-(1+\varepsilon)} = e^{-1} e^{-\varepsilon} < \frac{1}{e}$

$x \to -1-\varepsilon$ eksponent je

$-\sqrt{\frac{-2}{-\varepsilon}} = -\frac{\sqrt{2}}{\sqrt{\varepsilon}} \to -\infty \quad \varepsilon \to 0$

te $y \to e^{-\infty} = 0 \qquad f(1) = 1$



$D(f) = R \setminus [-1, 1)$

$R(f) = (0, \frac{1}{e}) \cup (\frac{1}{e}, 1] = (0, 1] \setminus \{\frac{1}{e}\}$

Inverzna: $\ln y = -\sqrt{\frac{x-1}{x+1}} \qquad \ln y \le 0 \qquad x \in (-\infty, -1) \cup [1, \infty)$

$\ln^2 y = \frac{x-1}{x+1} = 1 + \frac{-2}{x+1} \Rightarrow \frac{2}{x+1} = 1 - \ln^2 y \qquad x+1 = \frac{2}{1-\ln^2 y} \qquad x = -1 + \frac{2}{1-\ln^2 y}$

$x = f^{-1}(y) = -1 + \frac{2}{1-\ln^2 y} \qquad \ln^2 y \ne 1$ zbog $\ln y \le 0 \Rightarrow \ln y \ne -1$

tj $y \ne \frac{1}{e} \qquad f^{-1}(x) = -1 + \frac{2}{1-\ln^2 x}$

zbog $y > 0$ i $\ln y \le 0 \quad y \ne \frac{1}{e}$ imamo

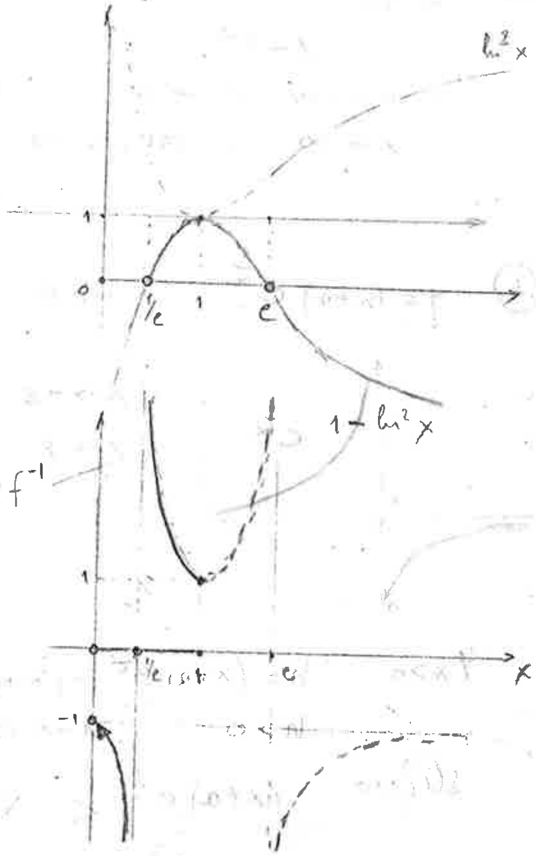
ukupno $0 < y \le 1$ i $y \ne \frac{1}{e} \quad (0,1] \setminus \{\frac{1}{e}\} = D(f^{-1})$

tj $R(f) = D(f^{-1}) = (0, 1] \setminus \{\frac{1}{e}\}$

$D(f) = R(f^{-1}) = R \setminus [-1, 1)$

$x \to \frac{1}{e}+ \quad f^{-1}(x) \to \infty \qquad x \to \varepsilon \quad f^{-1}(x) \to -1$

$x \to \frac{1}{e}- \quad f^{-1}(x) \to -\infty \qquad f^{-1}(1) = 1$

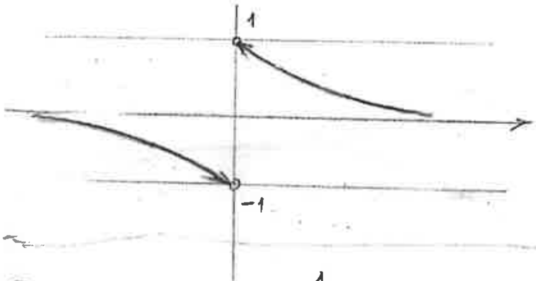


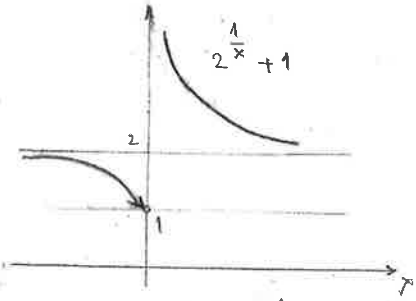
Monotonost: zbog $x_1 < x_2 \Rightarrow e^{x_1} < e^{x_2}$ i

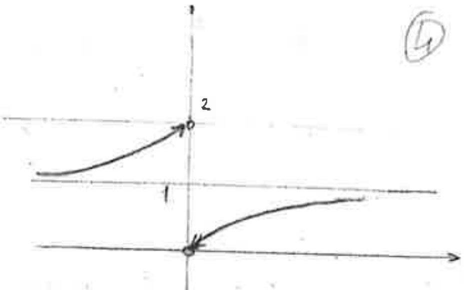
$e^{-x_2} < e^{-x_1}$ znači da $f(x)$ pada na $D(f)$

jer $\sqrt{\frac{x-1}{x+1}}$ raste na $R \setminus [-1, 1) = D(f)$

① $x \mapsto f(x) = \frac{2^{\frac{1}{x}}-1}{2^{\frac{1}{x}}+1} = 1 - \frac{2}{2^{\frac{1}{x}}+1}$





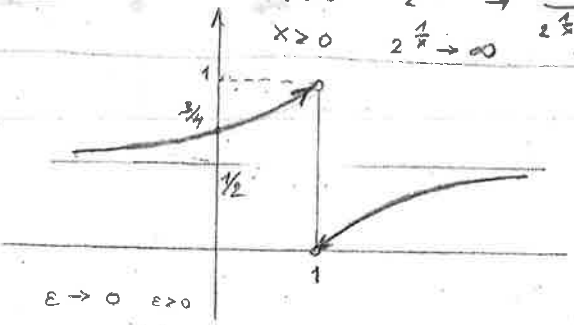


$x<0$ $\quad 2^{\frac{1}{x}} \to \frac{1}{2^{\frac{1}{x}}} \to 0 \quad x \to 0$

$x>0$ $\quad 2^{\frac{1}{x}} \to \infty \quad x \to 0$

$x \to \pm\infty$
$f(x) \to 0$

② $f(x) = \frac{1}{1+3^{-\frac{1}{1-x}}}$



$f(x) \to 0$
$x \to 1+ \quad 3^{-\frac{1}{1-x}} \to \infty$
$x \to 1+\varepsilon \quad 3^{\frac{1}{\varepsilon}} \quad \varepsilon \to 0$
$x \to \infty \quad 3^{-\frac{1}{-\infty}} \to 3^{\frac{1}{\infty}} \to 1$
$f(x) \to \frac{1}{2}$

$x \to 1-$ ili $x \to 1-\varepsilon \quad \varepsilon \to 0 \quad \varepsilon>0$

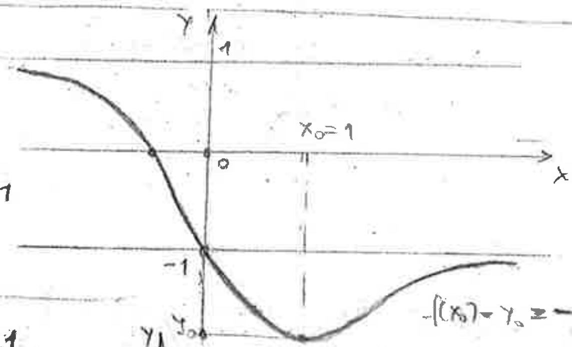
$3^{-\frac{1}{1-x}} \to 3^{-\frac{1}{\varepsilon}} \to \frac{1}{3^{\frac{1}{\varepsilon}}} \to \frac{1}{3^{\infty}} \to \frac{1}{\infty} \to 0 \qquad f(x) \to 1$

$x \to -\infty \qquad 3^{-\frac{1}{\infty}} \to \frac{1}{3^{\frac{1}{\infty}}} \to \frac{1}{3^0} \to 1 \qquad f(x) \to \frac{1}{2}$

③ $f(x) = \frac{x+e^x}{x-e^x} \qquad x=0 \quad f(x)=-1$

za $x \to -\infty \quad e^x \to 0 \quad f(x) \to \frac{x}{x} \to 1$

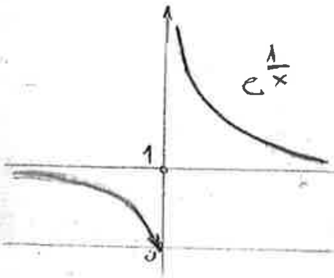
$x \to \infty \quad e^x$ brže teži ∞ nego x pa $f(x) \to -1$



$f(x_0)=y_0=-\frac{e+1}{e-1}$

④ $y=(x+a)e^{\frac{1}{x}} \qquad a>0 \qquad x \to \pm\infty \quad e^{\frac{1}{x}} \to 1$

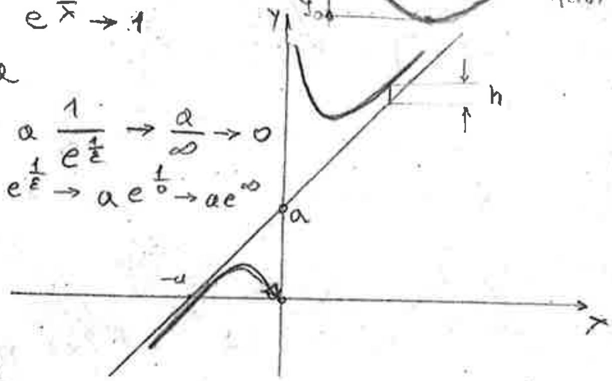
$y \to x+a$



$x \to -\varepsilon \quad \varepsilon \to 0 \quad y \to a\frac{1}{e^{\frac{1}{\varepsilon}}} \to \frac{a}{\infty} \to 0$

$x \to \varepsilon \quad \varepsilon \to 0 \quad y \to a e^{\frac{1}{\varepsilon}} \to a e^{\frac{1}{0}} \to a e^{\infty}$

$y \to \infty$



$\forall x>0 \quad h=(x+a)e^{\frac{1}{x}}-(x+a)$ zbog $e^{\frac{1}{x}}>1$ za $x>0$ i zbog $x+a>0 \quad (a>0)$

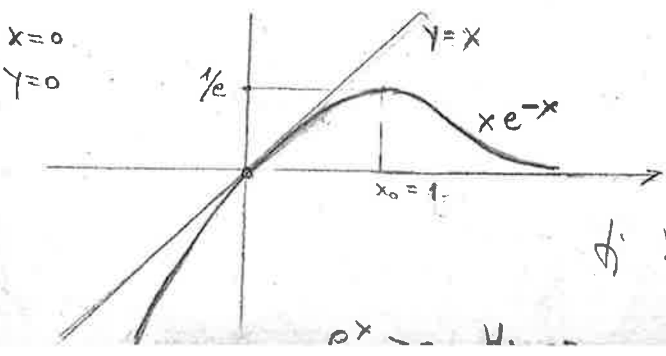
$h>0$ i $h \to 0$ za $x \to \infty$

slično $(x+a)e^{\frac{1}{x}} < x+a$ zbog $e^{\frac{1}{x}}<1$ za $x<0$ i za $x+a<0 \quad x<-a$

⑤ $y=xe^{-x} \qquad x \to -\infty \quad e^x \to 0$ a $e^{-x} \to \infty \quad y \to -\infty$

$x \to \infty \quad e^x$ brže teži ∞ nego x pa $y \to 0$

$x=0$
$y=0$

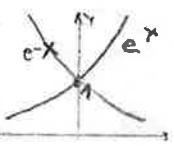


za $x>0 \Rightarrow e^x>1 \quad xe^x>x \Rightarrow xe^{-x}<x$

za $x<0 \quad -x>0 \quad e^{-x}>1 \Rightarrow$ zbog $-x>0$

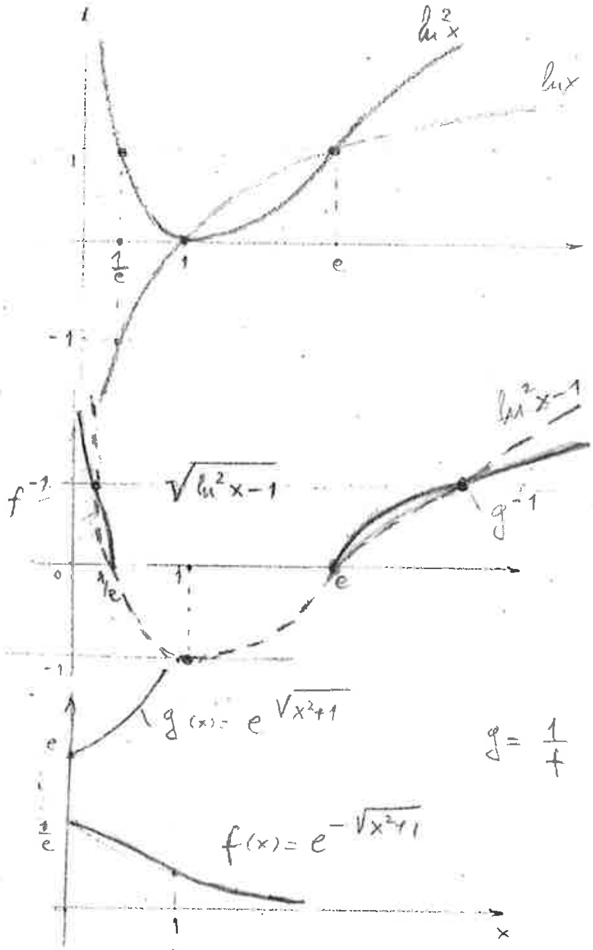
$-xe^{-x} > -x \Rightarrow xe^{-x}<x$

pa $\forall x \in \mathbb{R}$ vrijedi $\quad xe^{-x} \le x$



1) Skicirajte graf funkcije $f(x)=e^{-\sqrt{x^2+1}}$ $(x>0)$

Nađite f^{-1} i $\frac{1}{f}$



a) $\mathcal{D}(f)=\mathbb{R}^+$

$\ln y = -\sqrt{x^2+1}$ $\quad \ln y<0$

$\ln^2 y = x^2+1 \Rightarrow x^2 = \ln^2 y - 1$

$x=\sqrt{\ln^2 y-1}$ odakle je

$\ln^2 y - 1>0 \Rightarrow (\ln y+1)(\ln y-1)>0$

ovo je ispunjeno za $\ln y<-1$ ili $\ln y>1$

zbog $\ln y<0$ treba biti $\ln y<-1$ tj

$y<\frac{1}{e} \Rightarrow \mathcal{R}(f)=(0,\frac{1}{e})$

Skicirajmo graf f prema

$x\to\infty$ $\quad y\approx e^{-x}\to 0$

tj $y=0$ Horizontalna asimp.

za $x=0$ $\quad y=\frac{1}{e}$ (max.)

$y=f(x)=e^{-\sqrt{x^2+1}} \Rightarrow \ln y=-\sqrt{x^2+1}$ $\quad \ln y<0$

$y=f^{-1}(x)=\sqrt{\ln^2 x-1}$ $\quad \mathcal{D}(f^{-1})=(0,\frac{1}{e})=\mathcal{R}(f)$ $\quad \mathcal{D}(f)=\mathcal{R}(f^{-1})=\mathbb{R}^+$

b) Za $y=g(x)=e^{\sqrt{x^2+1}} \Rightarrow \ln y=\sqrt{x^2+1}$ $\quad \ln y>0$ te je $\ln y>1$

$y=g^{-1}(x)=\sqrt{\ln^2 x-1}$ $\quad \mathcal{D}(g^{-1})=(e,\infty)=\mathcal{R}(g)$ $\quad \mathcal{D}(g)=\mathcal{R}(g^{-1})=\mathbb{R}^+$

Vidimo da je $g(x)=\frac{1}{f(x)}$ pazi! $f^{-1}(x)\neq[f(x)]^{-1}$!!

Monotonost $f(x_1)-f(x_2)=e^{-\sqrt{x_1^2+1}}-e^{-\sqrt{x_2^2+1}}=\frac{e^{\sqrt{x_2^2+1}}-e^{\sqrt{x_1^2+1}}}{e^{\sqrt{x_1^2+1}}\,e^{\sqrt{x_2^2+1}}}$

iz monotonije $x_1<x_2 \Rightarrow e^{x_1}<e^{x_2}$ Dobivamo

za $\forall x_1, x_2>0 \Rightarrow x_1<x_2$ $\quad \sqrt{x_2^2+1}>\sqrt{x_1^2+1}$ odakle

$e^{\sqrt{x_2^2+1}}-e^{\sqrt{x_1^2+1}}>0$ tj $f(x_1)-f(x_2)>0$ ili $f(x_1)>f(x_2)$

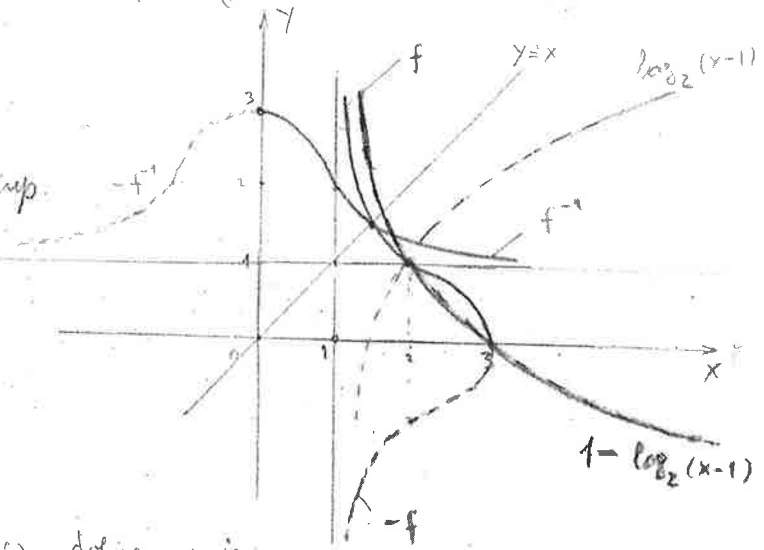
dakle f strogo pada na $\mathcal{D}(f)=\mathbb{R}^+$

(2) Inverzna funkcija f^{-1} dana je jednadžbom. Nađite f^{-1}

$y^2 - 1 + \log_2(x-1) = 0$

$x=3 \quad y=0$ Nultočka $x_0 = 3$

za $x \to 1$ $y \to \infty$ tj $x=1$ Vert. asimp.



$(\pm f(x))^2 = y^2 = 1 - \log_2(x-1)$

Kako je $y^2 \geq 0 \Rightarrow 1 - \log_2(x-1) \geq 0$

tj $\log_2(x-1) \leq 1$ i $x-1 > 0$

$x > 1$ i $x-1 \leq 2$ tj $x \leq 3$

te je $\mathcal{D}(f) = (1, 3]$ Sliku $\mathcal{R}(f)$ dobivamo iz

$\log_2(x-1) = 1-y^2 \qquad x-1 = 2^{1-y^2}$

uz $y \geq 0 \Rightarrow \quad x = f^{-1}(y) = 1 + 2^{1-y^2}$

$y = f^{-1}(x) = 1 + 2 \cdot 2^{-x^2} \quad (x \geq 0) \qquad \mathcal{D}(f^{-1}) = \mathbb{R}^+ = \mathcal{R}(f)$

$x \to \infty \quad y \to 1$ i $y = 3$ (max) za $x = 0$

Monotonost: $f(x_1) - f(x_2) = \sqrt{1-\log_2(x_1-1)} - \sqrt{1-\log_2(x_2-1)} =$

$= \dfrac{\log_2(x_2-1) - \log_2(x_1-1)}{\sqrt{} + \sqrt{}} = A \cdot \log_2 \dfrac{x_2-1}{x_1-1} > 0$ zbog $1 < x_1 < x_2$ jer je $\dfrac{x_2-1}{x_1-1} > 1$

za $\forall\, x_1, x_2 \in (1,3] = \mathcal{D}(f)$ što povlači $f(x_1) > f(x_2)$ tj f pada

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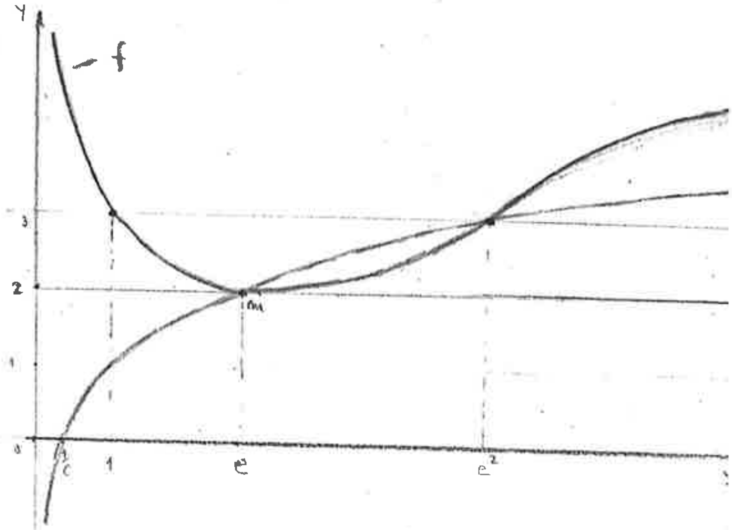
(3) Neka je $f(x) = (\ln x)^2 - 2\ln x + 3$. Nađite intervale monotonosti i na njima odgovarajuće inverzne funkcije f^{-1}?

$y = f(x) = (\ln x - 1)^2 + 2 \quad (x > 0)$



min za $\ln x - 1 = 0 \qquad y_{min} = 2$

$x = e \quad f(e) = 2 \qquad \mathcal{D}(f) = \mathbb{R}^+$

iz $(\ln x - 1)^2 = y - 2$ izlazi

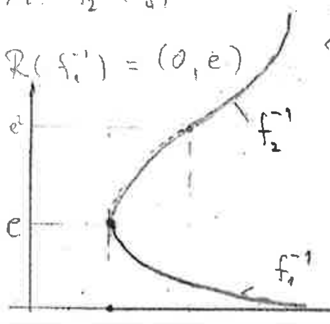
$\ln(x_{1,2}) = 1 \pm \sqrt{y-2}$ odavde $y \geq 2$

$x_{1,2} = e^{1 \pm \sqrt{y-2}} \qquad \mathcal{R}(f) = [2, \infty)$

$x_1 = f_1^{-1}(y) = e^{1-\sqrt{y-2}} \qquad \mathcal{D}(f_1^{-1}) = \mathcal{R}(f)$

$x_2 = f_2^{-1}(y) = e^{1+\sqrt{y-2}} \qquad \mathcal{D}(f_2^{-1}) = \mathcal{R}(f)$

$\mathcal{R}(f_1^{-1}) = (0, e) \qquad \mathcal{R}(f_2^{-1}) = (e, \infty)$



Vidimo da je $\mathcal{R}(f_1^{-1}) \cup \mathcal{R}(f_2^{-1}) = \mathcal{D}(f) = (0, \infty)$

$f_1^{-1}(x) = e^{1-\sqrt{x-2}} \qquad f_2^{-1}(x) = e^{1+\sqrt{x-2}}$

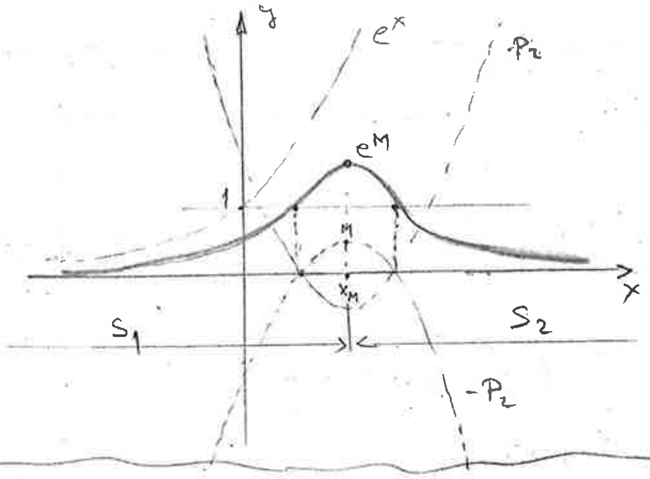
Monotonost: $f(x_1) - f(x_2) = (\ln x_1 - 1)^2 - (\ln x_2 - 1)^2 = (\ln x_1 - \ln x_2)(\ln x_1 + \ln x_2 - $

$= \ln \dfrac{x_1}{x_2} (\ln x_1 x_2 - 2)$ za $x_1, x_2 \in (0, e)$ uz $x_1 < x_2$ izlazi

$\dfrac{x_1}{x_2} < 1$ tj $f(x_1) - f(x_2) > 0$ i $f(x_1) > f(x_2)$ f pada

Zadana je funkcija $f(x) = e^{-(ax^2+bx+c)}$ $(a>0)$
Naďite $\mathcal{D}(f)$ i $\mathcal{R}(f)$. Odredite S_1 i S_2 tako da bude $S_1 \cup S_2 = \mathcal{D}(f)$, a $f|S_1$ i $f|S_2$ da imaju inverzne funkcije

$P_2(x) = ax^2+bx+c$



$S_1 = (-\infty, x_M)$ $\quad S_2 = (x_M, \infty)$

$\mathcal{R}(f) = (0, e^M]$

Zadana je funkcija $f: \mathbb{R} \to \mathcal{R}(f)$, $f: x \mapsto 2e^{-\sqrt[3]{x}} - 1$
Naďite f^{-1}, $\mathcal{R}(f)$, $\mathcal{D}(f^{-1})$ i $\mathcal{R}(f^{-1})$

(1) Ako je $S_k = e^{kx} + e^{-kx}$ izrazi $(e^x + e^{-x})^n$ $n \in N$ pomoću $S_n, S_{n-2}, \dots$

(3)

R: $(e^x+e^{-x})^n = e^{nx} + \binom{n}{1} e^{(n-1)x} e^{-x} + \binom{n}{2} e^{(n-2)x} e^{-2x} +$

$+ \binom{n}{3} e^{(n-3)x} e^{-3x} + \dots + \binom{n}{k} e^{(n-k)x} e^{-kx} + \dots + \binom{n}{n-1} e^{x} e^{-(n-1)x} + e^{-nx}$

Treba razlikovati slučajeve za n = parno (1) i n = neparno (2)

(I) n = parno u binomnom razvoju ima $n+1$ članova (neparno)

za n = parno imamo srednji član u razvoju kod $k = \frac{n}{2}$

srednji članovi: $\binom{n}{\frac{n}{2}-1}, \binom{n}{\frac{n}{2}}, \binom{n}{\frac{n}{2}+1}$ $n = 2p$ (parno) $\binom{n}{k} = \binom{n}{n-k}$

zbog $\binom{n}{\frac{n}{2}-1} = \binom{n}{\frac{n}{2}+1}$ i općenito $\binom{n}{\frac{n}{2}-k} = \binom{n}{\frac{n}{2}+k}$ $(k = 0, 1, \dots \frac{n}{2}-1)$

imamo ovaj razvoj dobiven spajanjem simetričnih članova:

$$(e^x+e^{-x})^n = (e^{nx} + e^{-nx}) + \binom{n}{1}\left[e^{(n-2)x} + e^{-(n-2)x}\right] +$$

$$+ \binom{n}{2}\left[e^{(n-4)x} + e^{-(n-4)x}\right] + \dots + \binom{n}{k}\left[e^{(n-2k)x} + e^{-(n-2k)x}\right] + \dots +$$

$$+ \binom{n}{\frac{n}{2}-1}\left[e^{2x} + e^{-2x}\right] + \binom{n}{\frac{n}{2}} \qquad \left(k = 0, 1, 2 \dots \frac{n}{2}-1\right)$$

zbog $S_{n-2k} = e^{(n-2k)x} + e^{-(n-2k)x}$ izlazi

$$(e^x+e^{-x})^n = S_n + \binom{n}{1} S_{n-2} + \binom{n}{2} S_{n-4} + \dots + \binom{n}{k} S_{n-2k} + \dots + \binom{n}{\frac{n}{2}-1} S_2 + \binom{n}{\frac{n}{2}}$$

(II) n = neparno sada imamo $n+1$ članova tj paran broj članova u ovom slučaju postoje dva srednja člana:

i to kod $k = \frac{n-1}{2}$ i $k = \frac{n+1}{2}$ sa koeficij. $\binom{n}{\frac{n-1}{2}}, \binom{n}{\frac{n+1}{2}}$ n = nepar.

zbog $\binom{n}{\frac{n-1}{2}} = \binom{n}{\frac{n+1}{2}}$ i općenito $\binom{n}{\frac{n-k}{2}} = \binom{n}{\frac{n+k}{2}}$ za $(k = 1, 3, 5, \dots n)$

Spajanjem simetričnih članova dobivamo ovaj razvoj

$$(e^x+e^{-x})^n = S_n + \binom{n}{1} S_{n-2} + \dots + \binom{n}{k} S_{n-2k} + \dots + \binom{n}{\frac{n-1}{2}} S_1$$

n = parno $(e^x+e^{-x})^n = \binom{n}{\frac{n}{2}} + \sum\limits_{k=0}^{\frac{n}{2}-1} \binom{n}{k} S_{n-2k} = \binom{2p}{p} + \sum\limits_{k=0}^{p-1} \binom{n}{k} S_{2(p-k)}$

$n = 2p$

n = neparno $(e^x+e^{-x})^n = \sum\limits_{k=0}^{\frac{n-1}{2}} \binom{n}{k} S_{n-2k} = \sum\limits_{k=0}^{p} \binom{2p+1}{k} S_{2(p-k)+1}$

napomena! $S_k = S_{-k}$! $n = 2p+1$

$$S_{2(p-k)} = e^{2(p-k)x} + e^{-2(p-k)x}$$