Hiperbolne funkcije

5. hiperbolne funkcije

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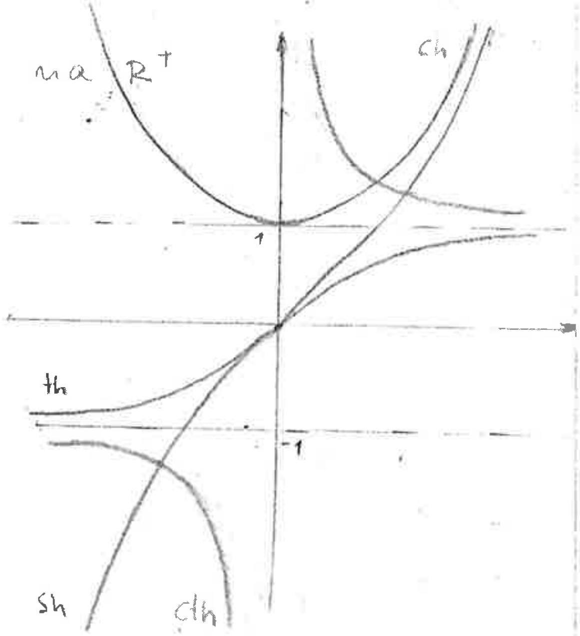
$$\operatorname{sh} x = \frac{1}{2}(e^x - e^{-x}) \qquad \operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \; ; \quad \operatorname{cth} x = \frac{\operatorname{ch} x}{\operatorname{sh} x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{ch} x = \frac{1}{2}(e^x + e^{-x})$$



Teorem: funkcije sh, ch i th strogo rastu na $\mathbb{R}^+$

Pokažimo da th strogo raste na $\mathbb{R}$:

pretpostavimo $h > 0$ i računajmo

1) $$\operatorname{th}(x+h) - \operatorname{th} x = \frac{e^{x+h} - e^{-x-h}}{e^{x+h} + e^{-x-h}} - \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$= \frac{2(e^h - e^{-h})}{(e^{x+h} + e^{-x-h})(e^x + e^{-x})} > 0 \quad \text{zbog } e^h > 1 \text{ i } e^{-h} < 1$$

a nazivnik je uvijek pozitivan

b) Funkcija ch strogo pada za $x \leq 0$, a strogo raste za $x > 0$

Neka je $x \geq 0$ i $h > 0$. Vrijedi

$$\operatorname{ch}(x+h) - \operatorname{ch} x = \frac{1}{2}\left(e^{x+h} + e^{-x-h} - e^x - e^{-x}\right)$$

$$= \frac{1}{2}\left[e^x(e^h - 1) - e^{-x-h}(e^h - 1)\right]$$

$$= \frac{1}{2}\left(e^x - e^{-(x+h)}\right)(e^h - 1) > 0$$ jer je za $x > 0$ i $h > 0$ $e^x > 1$, $e^{-(x+h)} < 1$ i $e^h > 1$. Za $x \leq 0$ imamo slično da je sada $e^x \leq 1$, $e^{-(x+h)} > 1$ te je lijeva strana negativna.

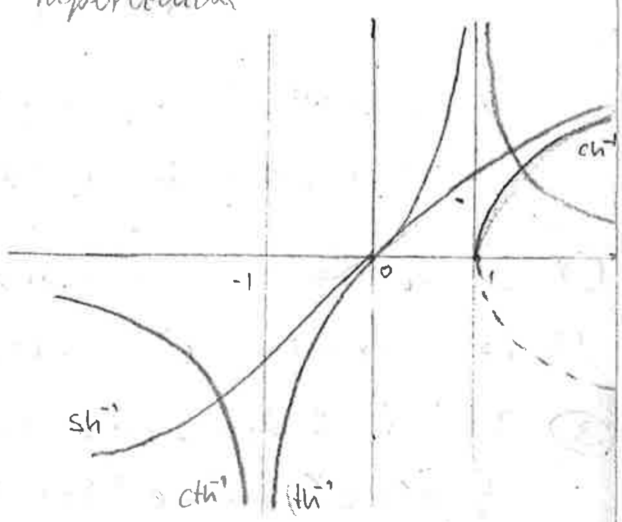
Area funkcije su inverzne funkcije hiperbolnih

$$\operatorname{sh}^{-1} x = \operatorname{arsh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\operatorname{ch}^{-1} x = \operatorname{arch} x = \ln\left(x + \sqrt{x^2 - 1}\right) \quad \text{za } x > 1$$

$$\operatorname{th}^{-1} x = \operatorname{arth} x = \ln\sqrt{\frac{1+x}{1-x}} \quad \text{za } |x| < 1$$

$$\operatorname{cth}^{-1} x = \operatorname{arcth} x = \ln\sqrt{\frac{x+1}{x-1}} \quad \text{za } |x| > 1$$



Osnovne formule za hiperbolne funkcije:

$$\operatorname{ch} 2x = \operatorname{ch}^2 x + \operatorname{sh}^2 x \qquad \operatorname{sh} 2x = 2 \operatorname{sh} x \operatorname{ch} x$$

$$1 = \operatorname{ch}^2 x - \operatorname{sh}^2 x \qquad \operatorname{ch} 2x = 2\operatorname{ch}^2 x - 1$$

① Pokaži da je $ch(x+y) = chx\, chy + shx\, shy$

iz $chx = \frac{1}{2}(e^x + e^{-x})$ i $shx = \frac{1}{2}(e^x - e^{-x})$ izlazi

(1) $chx + shx = e^x$ i $chx - shx = e^{-x}$ (2) a odavde npr množenjem dobivamo $(chx + shx)(chx - shx) = e^x \cdot e^{-x}$ odnosno važnu formulu

$$ch^2x - sh^2x = 1$$

Zbog (1) i (2) imamo $ch(x+y) + sh(x+y) = e^{x+y}$ odnosno

$ch(x+y) - sh(x+y) = e^{-(x+y)}$ i dalje je

$$e^x \cdot e^y = (chx + shx)(chy + shy) = e^{x+y} = ch(x+y) + sh(x+y) \quad \text{tj.}$$

(3) $ch(x+y) + sh(x+y) = (chx + shx)(chy + shy)$ slično je i

$$e^{-x}e^{-y} = (chx - shx)(chy - shy) = e^{-(x+y)} = ch(x+y) - sh(x+y) \quad \text{te je}$$

(4) $ch(x+y) - sh(x+y) = (chx - shx)(chy - shy)$ Zbrajanjem (3) i (4) izlazi

$$2\,ch(x+y) = (chx + shx)(chy + shy) + (chx - shx)(chy - shy)$$

$$= chx\,chy + \cancel{chx\,shy} + \cancel{shx\,chy} + shx\,shy + chx\,chy - \cancel{chx\,shy} - \cancel{shx\,chy} + shx\,shy$$

$$= 2\,chx\,chy + 2\,shx\,shy \Rightarrow ch(x+y) = chx\,chy + shx\,shy$$

Oduzimanjem (3) i (4) dobivamo slično i formulu za $sh(x+y)$ tj.

$$sh(x+y) = shx\,chy + chx\,shy$$

II NAČIN $sh(x+y) = \frac{1}{2}(e^{x+y} - e^{-(x+y)}) = \frac{1}{2}(e^x \cdot e^y - e^{-x}e^{-y})$

i iz $2\,shx = e^x - e^{-x}$, $2\,chx = e^x + e^{-x}$; $chx + shx = e^x$, $chx - shx = e^{-x}$ uvrstimo li ovo dobiva se

$$\frac{1}{2}\left[(chx + shx)(chy + shy) - (chx - shx)(chy - shy)\right] =$$

$$= \frac{1}{2}(2\,chx\,shy + 2\,shx\,chy) \Rightarrow sh(x+y) = shx\,chy + chx\,shy$$

② Dokaži da je $(chx + shx)^n = ch\,nx + sh\,nx$ $\quad n \in N$

•) iz $chx + shx = e^x \Rightarrow (chx + shx)^n = e^{nx} = ch\,nx + sh\,nx$

③ Izvedi $2\,shx\,shy = ch(x+y) - ch(x-y)$

$$\frac{1}{2}(e^x - e^{-x})(e^y - e^{-y}) = \frac{1}{2}(e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y}) =$$

$$= \frac{1}{2}\left[e^{x+y} + e^{-(x+y)} - (e^{x-y} + e^{-(x-y)})\right] = ch(x+y) - ch(x-y)$$

1. Izvedi $\operatorname{sh} x + \operatorname{sh} y = 2 \operatorname{sh} \frac{x+y}{2} \operatorname{ch} \frac{x-y}{2}$

$\frac{1}{2}\left(e^{x} - e^{-x} + e^{y} - e^{-y}\right) =$ uvedimo sada polovične argumente

$$\frac{1}{2}\left(e^{\frac{x+y}{2}+\frac{x-y}{2}} - e^{-\frac{x+y}{2}-\frac{x-y}{2}} + e^{\frac{x+y}{2}-\frac{x-y}{2}} - e^{-\frac{x+y}{2}+\frac{x-y}{2}}\right)$$

$$= \frac{1}{2}\left[e^{\frac{x+y}{2}}\left(e^{\frac{x-y}{2}} + e^{-\frac{x-y}{2}}\right) - e^{-\frac{x+y}{2}}\left(e^{\frac{x-y}{2}} + e^{-\frac{x-y}{2}}\right)\right] =$$

$$= \frac{1}{2}\left(e^{\frac{x+y}{2}} - e^{-\frac{x+y}{2}}\right)\left(e^{\frac{x-y}{2}} + e^{-\frac{x-y}{2}}\right) = 2 \operatorname{sh} \frac{x+y}{2} \operatorname{ch} \frac{x-y}{2}$$

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2. Dokaži da je $\dfrac{\operatorname{ch} 2x + \operatorname{ch} 4y}{\operatorname{sh} 2x + \operatorname{sh} 4y} = \operatorname{cth}(x+2y)$

prema $\operatorname{ch} x + \operatorname{ch} y = 2 \operatorname{ch} \frac{x+y}{2} \operatorname{ch} \frac{x-y}{2}$ dobivamo za naš

$\operatorname{sh} x + \operatorname{sh} y = 2 \operatorname{sh} \frac{x+y}{2} \operatorname{ch} \frac{x-y}{2}$ slučaj da je

$$\frac{\operatorname{ch}(x+2y)\cancel{\operatorname{ch}(x-2y)}}{\operatorname{sh}(x+2y)\cancel{\operatorname{ch}(x-2y)}} = \frac{\operatorname{ch}(x+2y)}{\operatorname{sh}(x+2y)} = \operatorname{cth}(x+2y) \ !$$

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3. Dokaži ako je $x+y+z=0$ da je $\operatorname{ch} 2x + \operatorname{ch} 2y + \operatorname{ch} 2z = 4 \operatorname{ch} x \operatorname{ch} y \operatorname{ch} z - 1$

$$\operatorname{ch} 2x + \operatorname{ch} 2y + \operatorname{ch} 2z = 2\operatorname{ch}(x+y)\operatorname{ch}(x-y) + \operatorname{ch} 2(x+y) =$$

Kako je $\operatorname{ch} 2\alpha = \operatorname{ch}^2\alpha + \operatorname{sh}^2\alpha$, $1 = \operatorname{ch}^2\alpha - \operatorname{sh}^2\alpha$ $\Rightarrow$ $\operatorname{ch} 2\alpha = 2\operatorname{ch}^2\alpha - 1$

$$= 2\operatorname{ch}(x+y)\operatorname{ch}(x-y) + 2\operatorname{ch}^2(x+y) - 1 = 2\operatorname{ch}(x+y)\left[\operatorname{ch}(x-y) + \operatorname{ch}(x+y)\right] - 1$$

$$= 2\operatorname{ch}(x+y)\, 2\operatorname{ch} x \operatorname{ch} y - 1 = 4 \operatorname{ch} x \operatorname{ch} y \operatorname{ch} z - 1$$

① Dokaži a) $sh(sh^{-1}x) = x$ c) $sh(ch^{-1}x) = \sqrt{x^2-1}$

b) $ch(ch^{-1}x) = x$ d) $ch(sh^{-1}x) = \sqrt{x^2+1}$

a) Stavimo $sh^{-1}x = y \Rightarrow x = sh\, y$ i $sh\, y = x$!

b) $ch^{-1}x = y$ tj. $x = ch\, y$ $ch\, y = x$ ili prema def.

$ch^{-1}x = \ln(x+\sqrt{x^2-1}) \Rightarrow ch(ch^{-1}x) = \frac{1}{2}(e^{ch^{-1}x} + e^{-ch^{-1}x})$

$= \frac{1}{2}\left(x+\sqrt{x^2-1} + \frac{1}{x+\sqrt{x^2-1}}\right) = \frac{1}{2}\left(x+\sqrt{x^2-1} + \frac{x-\sqrt{x^2-1}}{x^2-(x^2-1)}\right)$

$= \frac{1}{2}\left(x+\cancel{\sqrt{x^2-1}} + x - \cancel{\sqrt{x^2-1}}\right) = \frac{1}{2}\cdot 2x = x$

c) zbog $sh^2x = ch^2x - 1 \Rightarrow sh(ch^{-1}x) = \sqrt{[ch(ch^{-1}x)]^2-1}$

$= \sqrt{x^2-1}$ slično i

d) $ch(sh^{-1}x) = \sqrt{[sh(sh^{-1}x)]^2+1} = \sqrt{x^2+1}$

② Pretvori u produkt $sh^{-1}x + sh^{-1}y$

$sh(sh^{-1}x + sh^{-1}y) = sh(sh^{-1}x)\,ch(sh^{-1}y) + ch(sh^{-1}x)\,sh(sh^{-1}y) =$

$= x\sqrt{y^2+1} + y\sqrt{x^2+1}$ te je dakle sada inverzno

$sh^{-1}x + sh^{-1}y = sh^{-1}\left(x\sqrt{y^2+1} + y\sqrt{x^2+1}\right)$

Slično i $ch^{-1}x + ch^{-1}y \Rightarrow ch(ch^{-1}x + ch^{-1}y) =$

$ch(ch^{-1}x)\,ch(ch^{-1}y) + sh(ch^{-1}x)\,sh(ch^{-1}y) = xy + \sqrt{x^2-1}\sqrt{y^2-1}$ tj.

$ch^{-1}x + ch^{-1}y = ch^{-1}\left[xy + \sqrt{(x^2-1)(y^2-1)}\right]$

Specijalno za $x=y$ dobivamo formule:

$2\,sh^{-1}x = sh^{-1}(2x\sqrt{x^2+1})$ i $2\,ch^{-1}x = ch^{-1}(2x^2-1)$

Da je zaista $2x^2-1 = ch(2\,ch^{-1}x)$ možemo i provjeriti iz

$ch\,2\alpha = 2ch^2\alpha - 1 \Rightarrow ch(2ch^{-1}x) = 2[ch(ch^{-1}x)]^2 - 1 = 2x^2-1$!

① Dokaži $sh(ch^{-1}x) = \sqrt{x^2-1}$ $sh(sh^{-1}x) = x$

$ch(sh^{-1}x) = \sqrt{x^2+1}$ $ch(ch^{-1}x) = x$

(I) Stavimo $x = f(y) = sh\, y \Rightarrow y = f^{-1}(x) = sh^{-1}x$

$f(f^{-1}(x)) = (f \circ f^{-1})(x) = x \Rightarrow f = sh \quad f^{-1} = sh^{-1}$

ili direktno iz $sh(sh^{-1}x) = x \Rightarrow sh^{-1}x = sh^{-1}x$

$ch^{-1}x = y \quad x = ch\, y$ i $sh\, y = \sqrt{x^2-1}$ zbog $ch^2y - sh^2y = 1$

$\Rightarrow x^2 - (x^2-1) \equiv 1$ ili iz $x = ch\, y \Rightarrow sh\, y = \sqrt{x^2-1}$

(II) $sh(ch^{-1}x) = \sqrt{x^2-1}$ iz $ch^2(ch^{-1}x) - sh^2(ch^{-1}x) = 1$

$\Rightarrow sh(ch^{-1}x) = \sqrt{ch^2(ch^{-1}x) - 1}$ zbog $ch^2(ch^{-1}x) = [ch(ch^{-1}x)]^2 = x^2$

$sh(ch^{-1}x) = \sqrt{x^2-1}$ slično je i

$ch(sh^{-1}x) = \sqrt{[sh(sh^{-1}x)]^2+1} = \sqrt{x^2+1}$

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② $sh^{-1}x + sh^{-1}y$ pretvori u produkt

a) $sh(sh^{-1}x + sh^{-1}y) = sh(sh^{-1}x)\,ch(sh^{-1}y) + ch(sh^{-1}x)\,sh(sh^{-1}y) =$

$= x\sqrt{y^2+1} + y\sqrt{x^2+1} \Rightarrow sh^{-1}x + sh^{-1}y = sh^{-1}\left(x\sqrt{y^2+1} + y\sqrt{x^2+1}\right)$

Primjedba za $y = x \Rightarrow 2\,sh^{-1}x = sh^{-1}\,2x\sqrt{x^2+1}$

stavimo dalje da je $x \to \sqrt{x^2+1} \Rightarrow 2\,sh^{-1}\sqrt{x^2+1} = sh^{-1}\,2\sqrt{(x^2+1)(x^2+2)}$

b) $ch^{-1}x + ch^{-1}y \Rightarrow ch(ch^{-1}x + ch^{-1}y) =$

$ch(ch^{-1}x)\,ch(ch^{-1}y) + sh(ch^{-1}x)\,sh(ch^{-1}y) =$

$= xy + \sqrt{x^2-1}\sqrt{y^2-1} \Rightarrow$

$ch^{-1}x + ch^{-1}y = ch^{-1}\left[xy + \sqrt{(x^2-1)(y^2-1)}\right]$

za $y = x \Rightarrow 2\,ch^{-1}x = ch^{-1}(2x^2-1)$

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③ $sh(x+y) = \frac{1}{2}(e^{x+y} - e^{-x-y}) = \frac{1}{2}(e^x e^y - e^{-x} e^{-y}) =$

zbog $\left.\begin{array}{l} 2\,sh\,x = e^x - e^{-x} \\ 2\,ch\,x = e^x + e^{-x} \end{array}\right] \Rightarrow ch\,x \pm sh\,x = e^{\pm x}$

$= \frac{1}{2}\left[(ch\,x + sh\,x)(ch\,y + sh\,y) - (ch\,x - sh\,x)(ch\,y - sh\,y)\right] =$

$= \frac{1}{2}(\cancel{ch\,x\,ch\,y} + ch\,x\,sh\,y + sh\,x\,ch\,y + \cancel{sh\,x\,sh\,y} -$

$\cancel{-ch\,x\,ch\,y} + ch\,x\,sh\,y + sh\,x\,ch\,y - \cancel{sh\,x\,sh\,y})$

$= sh\,x\,ch\,y + ch\,x\,sh\,y$ što je i trebalo pokazati

(4) $\operatorname{th}(x+y) = \dfrac{\operatorname{th}x + \operatorname{th}y}{1 + \operatorname{th}x \operatorname{th}y} = \dfrac{\operatorname{sh}x \cdot \operatorname{ch}y + \operatorname{ch}x \operatorname{sh}y}{\operatorname{sh}x \operatorname{sh}y + \operatorname{ch}x \operatorname{ch}y} = \dfrac{\operatorname{sh}(x+y)}{\operatorname{ch}(x+y)} = \operatorname{th}(x+y)$

ili

$$\frac{\dfrac{e^x - e^{-x}}{e^x + e^{-x}} + \dfrac{e^y - e^{-y}}{e^y + e^{-y}}}{1 + \dfrac{(e^x - e^{-x})(e^y - e^{-y})}{(e^x + e^{-x})(e^y + e^{-y})}} = \frac{2e^x e^y + \cancel{e^x e^{-y}} - \cancel{e^{-x} e^{y}} - 2e^{-x} e^{-y}}{2e^x e^y + 2e^{-x} e^{-y}} =$$

$$= \frac{e^{x+y} - e^{-x-y}}{e^{xy} + e^{-x-y}} = \operatorname{th}(x+y)$$

1) Neka je T funkcionalni operator definisan pomoću jednakosti (6)

$$T f(x) = f(x+a)\, f(x+b) - f(x)\, f(x+a+b)$$

Dokaži 1° $\quad -T\,\mathrm{ch}\,x = T\,\mathrm{sh}\,x = \mathrm{sh}\,a\ \mathrm{sh}\,b$

2° $\quad T\sin x = T\cos x = \sin a \sin b$ $\qquad f = \mathrm{sh}$

1° $f(x) = \mathrm{sh}\,x \qquad T f(x) = \mathrm{sh}(x+a)\,\mathrm{sh}(x+b) - \mathrm{sh}\,x\ \mathrm{sh}(x+a+b)$

zbog $\mathrm{sh}\,x\,\mathrm{sh}\,y = \frac{1}{2}\left[\mathrm{ch}(x+y) - \mathrm{ch}(x-y)\right]$

$$Tf(x) = \frac{1}{2}\left[\cancel{\mathrm{ch}(2x+a+b)} - \mathrm{ch}(a-b) - \cancel{\mathrm{ch}(2x+a+b)} + \mathrm{ch}(a+b)\right]$$

$$T\,\mathrm{sh}\,x = \frac{1}{2}\left[\mathrm{ch}(a+b) - \mathrm{ch}(a-b)\right] = \mathrm{sh}\,a\ \mathrm{sh}\,b$$

$$T\,\mathrm{ch}\,x = \mathrm{ch}(x+a)\,\mathrm{ch}(x+b) - \mathrm{ch}\,x\ \mathrm{ch}(x+a+b)$$

$$= \frac{1}{2}\left[\cancel{\mathrm{ch}(2x+a+b)} + \mathrm{ch}(a-b) - \cancel{\mathrm{ch}(2x+a+b)} - \mathrm{ch}(a+b)\right]$$

$$= \frac{1}{2}\left[\mathrm{ch}(a-b) - \mathrm{ch}(a+b)\right]$$

$T\,\mathrm{ch}\,x = -\,\mathrm{sh}\,a\ \mathrm{sh}\,b \qquad$ tj. $\quad T\,\mathrm{sh}\,x = -T\,\mathrm{ch}\,x$

2° $\quad T\sin x = \sin(x+a)\sin(x+b) - \sin x \sin(x+a+b)$

zbog
$$\cos(x-y) = \cos x\cos y + \sin x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right] \qquad \cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$$

$$T\sin x = \frac{1}{2}\left[\cos(a-b) - \cancel{\cos(2x+a+b)} - \cos(a+b) + \cancel{\cos(2x+a+b)}\right]$$

$$= \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] = \sin a \sin b$$

$$T\cos x = \cos(x+a)\cos(x+b) - \cos x \cos(x+a+b)$$

$$= \frac{1}{2}\left[\cancel{\cos(2x+a+b)} + \cos(a-b) - \cancel{\cos(2x+a+b)} - \cos(a+b)\right]$$

$$= \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] = \sin a \sin b$$

tj. $\quad T\sin x = T\cos x = \sin a \sin b$.

① Dokaži $(\operatorname{ch} x \pm \operatorname{sh} x)^n = \operatorname{ch} nx \pm \operatorname{sh} nx$

$$\left.\begin{aligned} \operatorname{sh} x &= \tfrac{1}{2}(e^x - e^{-x}) \\ \operatorname{ch} x &= \tfrac{1}{2}(e^x + e^{-x}) \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \operatorname{ch} x + \operatorname{sh} x &= e^x \\ \operatorname{ch} x - \operatorname{sh} x &= e^{-x} \end{aligned}\right\} \Rightarrow \operatorname{ch} x \pm \operatorname{sh} x = e^{\pm x}$$

odavde je $(\operatorname{ch} x \pm \operatorname{sh} x)^n = e^{\pm nx}$, a kako je s druge strane

$$\left.\begin{aligned} \operatorname{sh} nx &= \tfrac{1}{2}(e^{nx} - e^{-nx}) \\ \operatorname{ch} nx &= \tfrac{1}{2}(e^{nx} + e^{-nx}) \end{aligned}\right\} \Rightarrow \operatorname{ch} nx \pm \operatorname{sh} nx = e^{\pm nx}$$

uspoređujući desne strane dobivamo da je $(\operatorname{ch} x \pm \operatorname{sh} x)^n = \operatorname{ch} nx \pm \operatorname{sh} nx$, što je i trebalo dokazati

2° $$\frac{\operatorname{ch} 2x + \operatorname{ch} 4y}{\operatorname{sh} 2x + \operatorname{sh} 4y} = \operatorname{cth}(x+2y)$$

kako je $\operatorname{ch} x + \operatorname{ch} y = 2 \operatorname{ch} \frac{x+y}{2} \operatorname{ch} \frac{x-y}{2}$

$\operatorname{sh} x + \operatorname{sh} y = 2 \operatorname{sh} \frac{x+y}{2} \operatorname{ch} \frac{x-y}{2}$ ili ako ovo primjenimo

izlazi $$\frac{\cancel{2} \operatorname{ch}(x+2y) \cancel{\operatorname{ch}(x-2y)}}{\cancel{2} \operatorname{sh}(x+2y) \cancel{\operatorname{ch}(x-2y)}} = \operatorname{cth}(x+2y)$$

3° Ako je $x+y+z=0$ dokaži formulu

$$\operatorname{ch} 2x + \operatorname{ch} 2y + \operatorname{ch} 2z = 4 \operatorname{ch} x \operatorname{ch} y \operatorname{ch} z - 1$$

Primjenimo li prethodne formule za zbroj ch dobivamo

$$2 \operatorname{ch}(x+y) \operatorname{ch}(x-y) + \operatorname{ch} 2(x+y) = 2 \operatorname{ch}(x+y) \operatorname{ch}(x-y) + 2\operatorname{ch}^2(x+y) - 1 =$$

$$= 2 \operatorname{ch}(x+y)[\operatorname{ch}(x+y) + \operatorname{ch}(x-y)] - 1 = 2 \operatorname{ch}(x+y) \cdot 2 \operatorname{ch} x \operatorname{ch} y - 1$$

te je konačno $\operatorname{ch} 2x + \operatorname{ch} 2y + \operatorname{ch} 2z = 4 \operatorname{ch} x \operatorname{ch} y \operatorname{ch} z - 1$

što je i trebalo pokazati!

① Izračunaj x iz $ch\, x = 1{,}2$

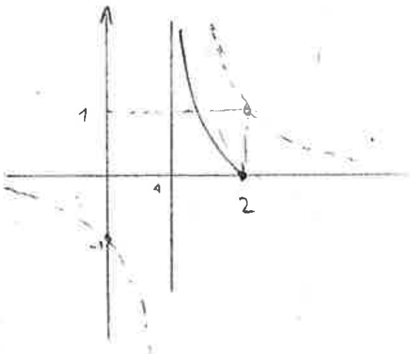
$y = ch\, x \Rightarrow x = ch^{-1} y = \ln\left(y + \sqrt{y^2 - 1}\right) = \ln\left(1{,}2 + \sqrt{1{,}44 - 1}\right)$

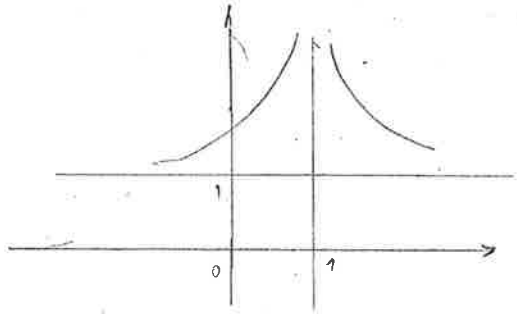
$x = \ln\left(1{,}2 + \sqrt{0{,}44}\right) = \ln\left(1{,}2 + 0{,}6633\right) = \ln\left(1{,}8633\right)$

kako je $\ln a = \frac{1}{\log e} \log a = \frac{1}{0{,}4343} \log a = 2{,}3 \log a$ te je

$\ln 1{,}863 = 2{,}3 \cdot 0{,}27 = 0{,}6215\ldots \qquad x = 0{,}62\ldots$

② Nacrtaj $y = Arch \frac{1}{x-1}$ $\qquad y = ch \frac{1}{x-1}$





Dokaži ③ $sh(ch^{-1}x) = \sqrt{x^2-1}$ ① $sh(sh^{-1}x) = x$

④ $ch(sh^{-1}x) = \sqrt{x^2+1}$ ② $ch(ch^{-1}x) = x$

① $sh^{-1}x = y \Rightarrow x = sh\,y$ ili $sh\,y = x$

② $ch^{-1}x = y \Rightarrow x = ch\,y$ ili $ch\,y = 1$

ili $sh(sh^{-1}x) = y$ treba biti $x = y$

Dokaz: $sh^{-1}x = u \Rightarrow x = sh\,u$ i jednake i

$sh\,u = y$ suprotne strane ti $\Rightarrow x = y$

③ $ch^{-1}x = y \Rightarrow x = ch\,y \Rightarrow sh\,y = \sqrt{x^2-1}$

$\boxed{ch^2 y - sh^2 y = 1}$ ti $sh(ch^{-1}x) = \sqrt{x^2-1}$

④ $sh^{-1}x = y$ $x = sh\,y \Rightarrow ch\,y = \sqrt{1+x^2}$

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ili dok $ch^2(sh^{-1}x) - sh^2(sh^{-1}x) = 1$

$$\Rightarrow ch(sh^{-1}x) = \sqrt{1+[sh(sh^{-1}x)]^2} = \sqrt{1+x^2}$$

$$ch^2(ch^{-1}x) - sh^2(ch^{-1}x) = 1$$

$$sh(ch^{-1}x) = \sqrt{[ch(ch^{-1}x)]^2-1} = \sqrt{x^2-1}$$

$sh^{-1}x + sh^{-1}y =$ pretvori u produkt!

$$sh(sh^{-1}x + sh^{-1}y) = sh(sh^{-1}x)\cdot ch(sh^{-1}y) +$$

$$+ ch(sh^{-1}x)\, sh(sh^{-1}y) = x\sqrt{y^2+1} + y\sqrt{x^2+1}$$

$$\Rightarrow sh^{-1}x + sh^{-1}y = sh^{-1}\left(x\sqrt{y^2+1} + y\sqrt{x^2+1}\right)$$

za $x = y \Rightarrow \underline{2\,sh^{-1}x = sh^{-1}\, 2x\sqrt{x^2+1}}$

Vježba $ch^{-1}x + ch^{-1}y = ?$

$$ch(ch^{-1}x + ch^{-1}y) = \cancel{x\,y} + sh(ch^{-1}x)\cdot sh(ch^{-1}y) =$$

$$xy + \sqrt{(x^2-1)(y^2-1)} \Rightarrow$$

$$ch^{-1}x + ch^{-1}y = ch^{-1}\left[xy + \sqrt{(x^2-1)(y^2-1)}\right]$$

Uzmimo $x = y \Rightarrow$

Rešeni zadevi:

1) $y = \frac{1}{\ln x} \qquad y' = -\frac{1}{x \ln^2 x}$

2) $y = \frac{\ln x}{x} \qquad y' = \frac{\frac{1}{x} \cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$

3) $y = \ln(\operatorname{sh} x) \qquad y' = \frac{1}{\operatorname{sh} x} \cdot \operatorname{ch} x = \operatorname{cth} x$

$y = \operatorname{sh}(\operatorname{sh}^{-1} x) \qquad y' = \frac{\operatorname{ch}(\operatorname{sh}^{-1} x) \cdot}{\sqrt{x^2+1}} = \frac{\sqrt{y^2+1}}{\sqrt{x^2+1}} = 1$

$\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1$

$\operatorname{sh}^2 x = 1 + \operatorname{ch}^2 x \qquad \operatorname{ch} x = \sqrt{\operatorname{sh}^2 x + 1} = \sqrt{y^2+1}$

$\operatorname{sh}^{-1} x = \operatorname{sh}^{-1} y \Rightarrow x = y$

$\left.\begin{array}{ll} \operatorname{sh}^{-1} x = u \Rightarrow & x = \operatorname{sh} u \\ y = \operatorname{sh} u & y = \operatorname{sh} u \end{array}\right\} \Rightarrow x = y$

$y = \operatorname{sh}^{-1}(\operatorname{sh} x) \qquad y' = \frac{\operatorname{ch} x}{\sqrt{\operatorname{sh}^2 x + 1}} = \frac{\operatorname{ch} x}{\operatorname{ch} x} = 1$

$\operatorname{ch}(\operatorname{sh}^{-1} x) = ?$

$\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1 \qquad$ ali $\qquad \operatorname{ch}^2(\operatorname{sh}^{-1} x) - \operatorname{sh}^2(\operatorname{sh}^{-1} x) = 1$

$y = \operatorname{sh}(\operatorname{sh}^{-1} x) \qquad \operatorname{ch}(\operatorname{sh}^{-1} x) = \sqrt{1 + y^2}$

$$\boxed{\operatorname{ch}(\operatorname{sh}^{-1} x) = \sqrt{1 + x^2}}$$

~~$\operatorname{sh}^{-1} x + \operatorname{sh}^{-1} y = \operatorname{sh}^{-1}\left(x\sqrt{1+y^2} + y\sqrt{x^2+1}\right)$~~

~~R: $\operatorname{sh}(\operatorname{sh}^{-1} x + \operatorname{sh}^{-1} y) = \operatorname{sh}(\operatorname{sh}^{-1} x)\operatorname{ch}(\operatorname{sh}^{-1} y) +$~~

~~$+ \operatorname{ch}(\operatorname{sh}^{-1} x) \cdot \operatorname{sh}(\operatorname{sh}^{-1} y) =$~~

$\operatorname{sh}(x+y) = \frac{1}{2}\left(e^{x+y} - e^{-x-y}\right) = \frac{1}{2}\left(e^x \cdot e^y - e^{-x} \cdot e^{-y}\right)$

$\operatorname{sh} x = e^x - e^{-x} \qquad = \frac{1}{2}\left[(\operatorname{ch} x + \operatorname{sh} x)(\operatorname{ch} y + \operatorname{sh} y) -\right.$

$\underline{\operatorname{ch} x = e^x + e^{-x}} \qquad \left. - (\operatorname{ch} x - \operatorname{sh} x)(\operatorname{ch} y - \operatorname{sh} y)\right]$

$\operatorname{ch} x \pm \operatorname{sh} x = e^{\pm x}$

zmnožimo