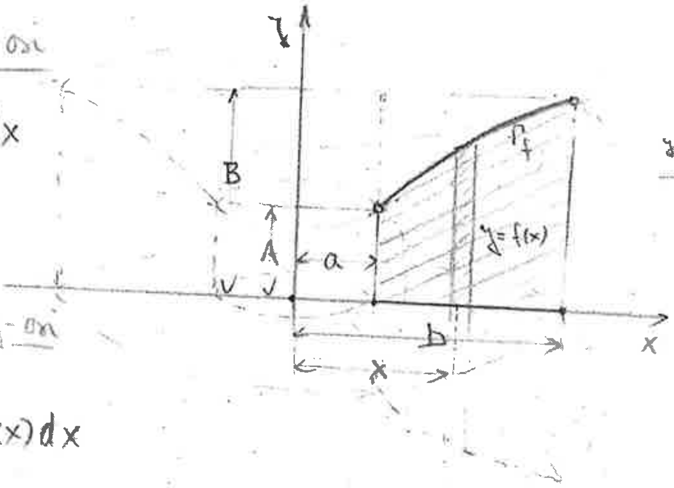
3. Volumen

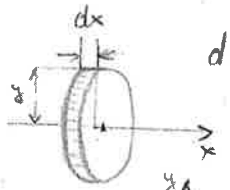
Volumen

Neka je zadana ploha $P = \{(x,y) : 0 \le a \le x \le b,\ 0 \le y \le f(x)\}$
koja rotira 1. oko x-osi
2. oko y-osi : odredi volumene tijela nastalih tim rotacijama

1. Rotacija oko x-osi

$$V_x = \pi \int_a^b [f(x)]^2 dx$$

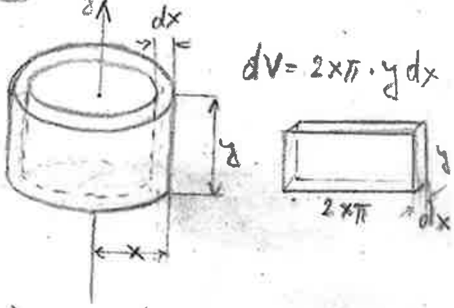




$$dV = y^2 \pi dx$$

2. Rotacija oko y-osi

$$V_y = 2\pi \int_a^b x f(x) dx$$



$$dV = 2x\pi \cdot y\, dx$$

Dokažimo pokazati formulu za V_y i na ovaj način

$$V_y = b^2 \pi B - a^2 \pi A - \pi \int_A^B x^2 dy$$ izračunajmo $\int_A^B x^2 dy$ posebno

Neka je $y = f(x) \Rightarrow x = f^{-1}(y)$

$$\int_A^B x^2 dy = \int_A^B [f^{-1}(y)]^2 dy = y[f^{-1}(y)]^2 \Big|_A^B - 2\int_A^B y f^{-1}(y)(f^{-1})'(y)\, dy = B[f^{-1}(B)]^2 - A[f^{-1}(A)]^2 - I_1$$

$I_1 = 2\int_A^B y f^{-1}(y)(f^{-1})'(y)\, dy$ izvršimo sada integraciju po x tj. uvedimo novu varijablu tj. stavimo $y = f(x) \Rightarrow$ za $y = A \Rightarrow x = a$
tada je i $dy = f'(x) dx$ i služimo $x = f^{-1}(y)$ $y = B \Rightarrow x = b$

zbog $(f^{-1})'(y) = \dfrac{1}{f'(f^{-1}(y))} = \dfrac{1}{f'(x)}$ uvrstimo sve u integral

$$I_1 = 2\int_a^b x f(x) \frac{1}{f'(x)} \cdot f'(x) dx = 2\int_a^b x f(x) dx$$ te je konačno

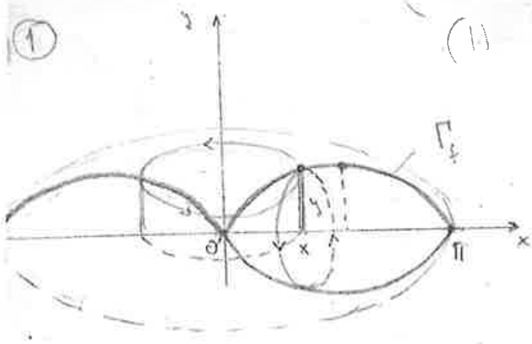
$$V_y = \cancel{b^2 \pi B} - \cancel{a^2 \pi A} - \cancel{b^2 B \pi} + \cancel{a^2 A \pi} + 2\pi \int_a^b x f(x) dx$$

$$V_y = 2\pi \int_a^b x f(x) dx$$

OPLOŠJE rotacione površine:

$$P = 2\pi \int_a^b y \sqrt{1 + y'^2}\, dx$$

(1)



(I) $V_x = \pi \int_a^b [f(x)]^2\,dx$, $\quad V_y = 2\pi \int_a^b x\, f(x)\,dx$

$x \mapsto \sin x \qquad x \in [0, \pi]$

$$V_x = \pi \int_0^\pi \sin^2 x\,dx = \frac{\pi}{2}\int_0^\pi (1-\cos 2x)\,dx = \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi$$

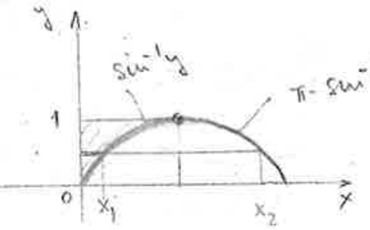
$$V_x = \frac{\pi}{2}(\pi - 0) \Rightarrow \boxed{V_x = \frac{\pi^2}{2}}$$

(II) Rotacijom oko Y-osi $V_y = 2\pi \int_0^\pi x \sin x\,dx = 2\pi\left[-x\cos x + \sin x\right]_0^\pi$

$V_y = 2\pi(\pi - 0) \quad \boxed{V_y = 2\pi^2}$

$x_2(y) = \pi - \sin^{-1} y$
$x_1(y) = \sin^{-1} y$

II način za rotaciju oko Y-osi prema



$$V_y = \pi \int_0^1 [x_2(y)]^2\,dy - \pi\int_0^1 [x_1(y)]^2\,dy$$

$$V_y = \pi \int_0^1 \left[(\pi - \arcsin y)^2 - (\arcsin y)^2\right]dy$$

$$V_y = \pi \int_0^1 (\pi - \arcsin y - \arcsin y)(\pi - \cancel{\arcsin y} + \cancel{\arcsin y})\,dy$$

$$V_y = \pi^2 \int_0^1 (\pi - 2\arcsin y)\,dy = \pi^3 - 2\pi^2 \int_0^1 \arcsin y\,dy \qquad y=0 \quad t=0$$

$I_1 = \int_0^1 \arcsin y\,dy \quad \arcsin y = t \quad y = \sin t \quad dy = \cos t\,dt \qquad y=1 \quad t = \frac{\pi}{2}$

$$I_1 = \int_0^{\pi/2} t\cos t\,dt = t\sin t + \cos t\Big|_0^{\pi/2} = \frac{\pi}{2} - 1 \qquad \boxed{V_y = 2\pi^2}$$

$$V_y = \pi^3 - 2\pi^2\left(\frac{\pi}{2} - 1\right) = \cancel{\pi^3} - \cancel{\pi^3} + 2\pi^2 \Rightarrow$$

$$V_1 = \pi \int_0^1 (\pi - \arcsin y)^2\,dy = \pi \int_\pi^{\pi/2} u^2 \cos u\,du = \pi\left(u^2 \sin u\Big|_\pi^{\pi/2} - 2\int_\pi^{\pi/2} u \sin u\,du\right)$$

$$V_1 = \pi\left[\frac{\pi^2}{4} + (2u\cos u - 2\sin u)\Big|_\pi^{\pi/2}\right] =$$

$\pi - \arcsin y = u$
$\arcsin y = \pi - u \qquad y = \sin(\pi - u)$

$$V_1 = \pi\left(\frac{\pi^2}{4} - 2 + 2\pi\right) = \frac{\pi^3}{4} + 2\pi^2 - 2\pi$$

$y = 0 \quad u = \pi \qquad dy = \cos u\,du$
$y = 1 \quad u = \frac{\pi}{2}$

$$V_2 = \pi \int_0^1 (\arcsin y)^2\,dy = \pi \int_0^{\pi/2} u^2 \cos u\,du = \pi\left[\frac{\pi^2}{4} + (2u\cos u - 2\sin u)\Big|_0^{\pi/2}\right]$$

$$V_2 = \pi\left(\frac{\pi^2}{4} - 2\right) = \frac{\pi^3}{4} - 2\pi \qquad V_y = V_1 - V_2 = 2\pi^2$$

[illegible]: računanje integrala za V_1

$$V_1 = \pi \int_0^1 (\pi - \arcsin y)^2\,dy = \pi \int_0^1 \left[\pi^2 - 2\pi \arcsin y + (\arcsin y)^2\right]dy =$$

$$V_1 = \pi\left[\pi^2 - 2\pi \int_0^1 \arcsin y\,dy + \int_0^1 (\arcsin y)^2\,dy\right] = \pi\left[\pi^2 - 2\pi\left(\frac{\pi}{2} - 1\right) + \left(\frac{\pi^2}{4} - 2\right)\right]$$

$$V_1 = \pi\left(\cancel{\pi^2} - \cancel{\pi^2} + 2\pi + \frac{\pi^2}{4} - 2\right) = \pi\left(\frac{\pi^2}{4} + 2\pi - 2\right) = \frac{\pi^3}{4} + 2\pi^2 - 2\pi$$

Površina astroide $x^{2/3}+y^{2/3}=a^{2/3}$

parametarski $x=a\cos^3 t$, $y=a\sin^3 t$, $dx=-3a\cos^2 t\sin t\,dt$

$$P=4\cdot\int_{\frac{\pi}{2}}^{0} a\sin^3 t\,(-3a\cos^2 t\sin t)\,dt=4\cdot 3a^2\int_0^{\frac{\pi}{2}}\sin^4 t\cos^2 t\,dt$$

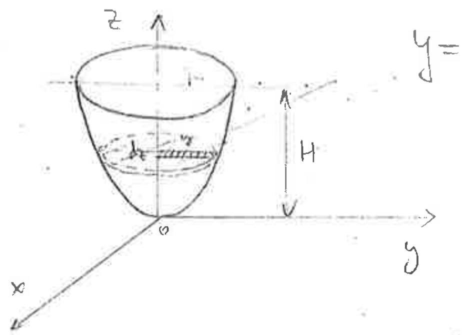
$$=\frac{3}{4}a^2\int_0^{\frac{\pi}{2}} 2\sin^2 2t\cdot 2\sin^2 t\,dt=\frac{3}{4}a^2\int_0^{\frac{\pi}{2}}(1-\cos 4t)(1-\cos 2t)\,dt=$$

$$=\frac{3}{4}a^2\int_0^{\frac{\pi}{2}}(1-\cos 2t-\cos 4t+\cos 2t\cos 4t)\,dt=\frac{3}{4}a^2\cdot\frac{\pi}{2}=\frac{3a^2\pi}{8}$$

OBILAZAK ZATVORENE KRIVULJE U NEGATIVNOM SMISLU

Volumen rotacionog paraboloida

(a)

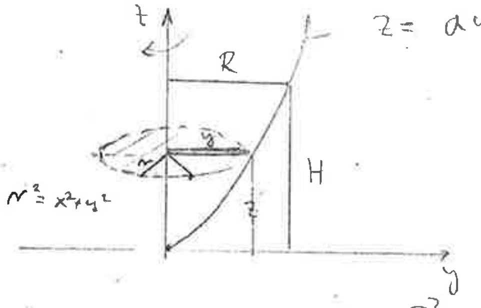


$y=\sqrt{z}$ $\quad dV=y^2\pi\,dz$ $\quad V=\pi\int_0^H y^2\,dz=$

$$V=\pi\int_0^H z\,dz=\pi\frac{z^2}{2}\Big|_0^H=\frac{H^2\pi}{2}$$

$Q(z)=y^2\pi$ poprečni presjek $(R=1)$

(b) Izračunaj volumen rotacionog paraboloida visine H i polumjera baze R na toj visini



$z=ay^2 \Rightarrow H=aR^2 \Rightarrow a=\frac{H}{R^2}$ tj.

$z=\frac{H}{R^2}y^2$. Rotacijom oko z-osi izlazi

$y^2=\frac{R^2}{H}z$ nadomjestimo

umjesto y^2 pišemo x^2+y^2

izlazi $x^2+y^2=\frac{R^2}{H}z$ $\quad Q(z)$ poprečni presjek na visini z

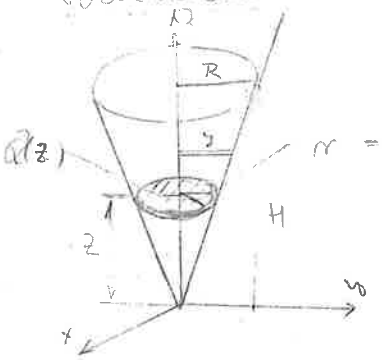
$V=\int_0^H Q(z)\,dz$ uz $Q(z)=r^2\pi=(x^2+y^2)\pi=\frac{\pi R^2}{H}z$

$$V=\int_0^H\frac{\pi R^2}{H}z\,dz=\frac{\pi R^2}{H}\int_0^H z\,dz=\frac{\pi R^2}{H}\cdot\frac{H^2}{2}=\frac{\pi}{2}R^2H$$

Direktno rotacijom krivulje $y=R\sqrt{\frac{z}{H}}$ oko z-osi

$$V=\pi\int_0^H y^2\,dz=\frac{R^2\pi}{H}\int_0^H z\,dz=\frac{R^2\pi}{2}H$$

Volumen stošca visine H i polumjera baze R na toj visini



$r = \sqrt{x^2+y^2}$ $\qquad \dfrac{r}{R} = \dfrac{z}{H} \qquad z = \dfrac{H}{R} r = \dfrac{H}{R}\sqrt{x^2+y^2}$

dobivamo $z^2 = \dfrac{H^2}{R^2}(x^2+y^2)$ ili

$x^2+y^2 = \dfrac{R^2}{H^2} z^2$ --- stožac

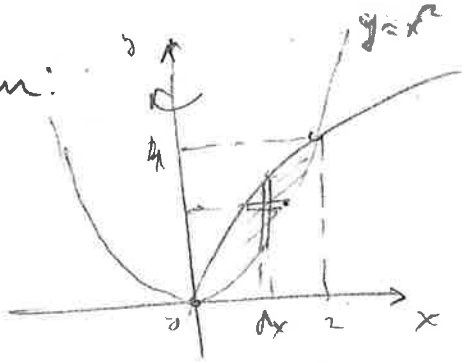
Volumen: pravac $y = \dfrac{R}{H} z$ rotira oko z-osi tj

$$V = \pi \int_0^H \left(\frac{R}{H} z\right)^2 dz = \frac{R^2\pi}{H^2}\int_0^H z^2\,dz = \frac{R^2\pi}{H^2}\cdot\frac{H^3}{3} = \frac{\pi}{3}R^2H$$

Površina poprečnog presjeka $Q(z) = r^2\pi = (x^2+y^2)\pi = \dfrac{R^2}{H^2} z^2 \cdot \pi$

$$V = \int_0^H Q(z)\,dz = \frac{R^2}{H^2}\pi\int_0^H z^2\,dz$$, jasno ista formula.

1 Volumen:



$y^2 = 8x$

$V_y = \frac{24}{5}\pi$

$$V_y = \pi \int_0^4 \left(y - \frac{y^4}{64}\right) dy = \frac{24}{5}\pi$$

ili

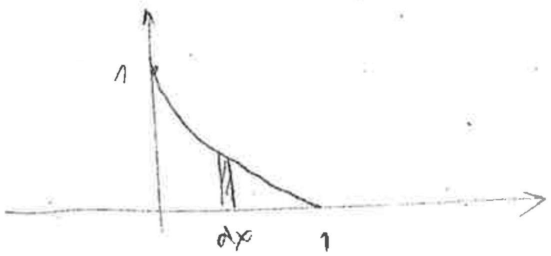
$$\pi \int_0^2 2x\left(\sqrt{8x} - x^2\right) dx = \pi \int_0^2 \left(4x^{\frac{3}{2}} - 2x^3\right) dx = \pi\left(\frac{8\sqrt{2}}{5}x^{\frac{5}{2}} - \frac{1}{2}x^4\right)\Big|_0^2$$

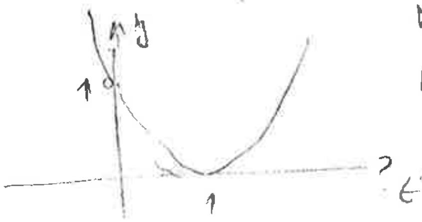
$$= \pi\left(\frac{64}{5} - 8\right) = \frac{24}{5}\pi$$

2.) $\sqrt{x} + \sqrt{y} = 1$ $\quad x = t^2$ $\quad x \geq 0 \quad y \geq 0$

$y = t^2 - 2t + 1$

$V_x = \frac{\pi}{15}$





$\sqrt{y} = 1 - \sqrt{x} \geq 0$

$\sqrt{x} = 1 - \sqrt{y} \geq 0$

$\sqrt{x} \leq 1$

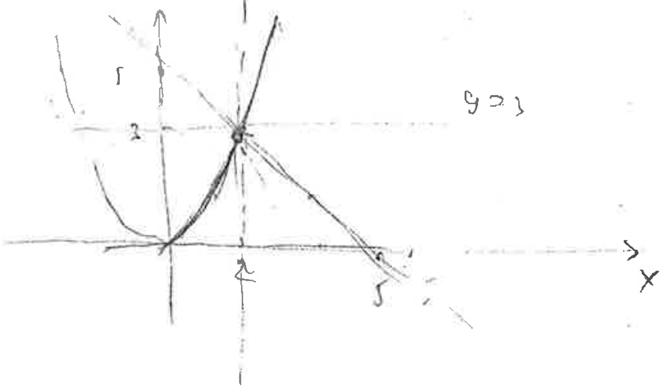
$0 < x \leq 1$!

$0 < y \leq 1$!

3) Volumen rotacijom oko a) $x=2$

b) $y=3$

Plohe omeđene su $y = \frac{3}{4}x^2$ $\quad y = -x+5$ i $\quad y = 0$



4.)

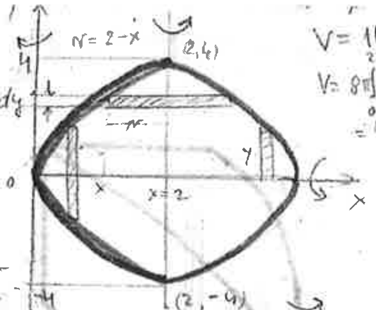
Volumeni rotacionih tijela

1) Naoti volumen što ga čini parabola $y^2 = 8x$ rotacijom oko $x = 2$

$dV = r^2\pi\, dy = (2-x)^2 \pi\, dy = \pi\left(2 - \frac{y^2}{8}\right)^2 dy \qquad V = \pi\int_{-4}^{+4}\left(2 - \frac{y^2}{8}\right)^2 dy$

$V = 2\pi \int_0^4 \left(4 - \frac{y^2}{2} + \frac{y^4}{64}\right) dy = 2\pi\left(4y - \frac{y^3}{6} + \frac{y^5}{5\cdot 64}\right)\Big|_0^4 =$

$= 2\pi\left(16 - \frac{2\cdot 16}{3} + \frac{16}{5}\right) = \frac{32\pi}{15}(15 - 10 + 3) = \frac{32\cdot 8\pi}{15} = \underline{\underline{\frac{256}{15}\pi}}$



$V = 16\pi$

$V = 8\pi\int_0^2 x\, dx$

$= 4x^2\pi\Big|_0^2$

2) Naoti volumen što ga čini ploha omeđena parabolom $y^2 = 8x$ i pravcem $x = 2$ rotacijom oko y-osi.

$dV = 2^2\pi\, dy - x^2\pi\, dy = \pi(4 - x^2)\, dy$

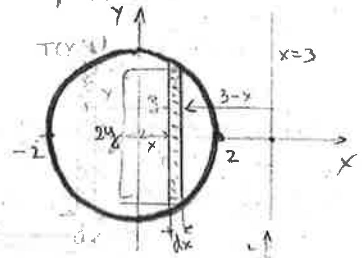
$V = \pi\int_{-4}^{4}(4 - x^2)\, dy = 2\pi\int_0^4 (4 - x^2)\, dy = 2\pi\int_0^4\left(4 - \frac{y^4}{64}\right) dy = \underline{\underline{\frac{128}{5}\pi}}$

$= 2\cdot 2\pi\int_0^2 x\cdot\sqrt{8x}\, dx$

$dV = 2\pi(2-x)\cdot 2y\, dx$

3) Naoti volumen torusa kojeg čini krug $x^2 + y^2 = 4$ rotacijom oko pravca $x = 3$

(I)



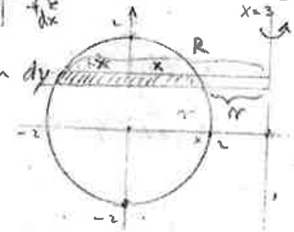
$dV = 2\pi(3-x)\cdot 2y\, dx \qquad y^2 = 4 - x^2 \Rightarrow y = \sqrt{4-x^2}$

$dV = 4\pi(3-x)\cdot\sqrt{4-x^2}\, dx \qquad V = 4\pi\int_{-2}^{2}(3-x)\sqrt{4-x^2}\, dx =$

$= 12\pi\int_{-2}^{2}\sqrt{4-x^2}\, dx - 4\pi\int_{-2}^{2} x\sqrt{4-x^2}\, dx =$

$= \left[12\pi\left(\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right) + \frac{4\pi}{3}(4-x^2)^{3/2}\right]_{-2}^{2} = \underline{\underline{24\pi^2}}$

(II) Način dy



volumen elementarnog prstena dV je

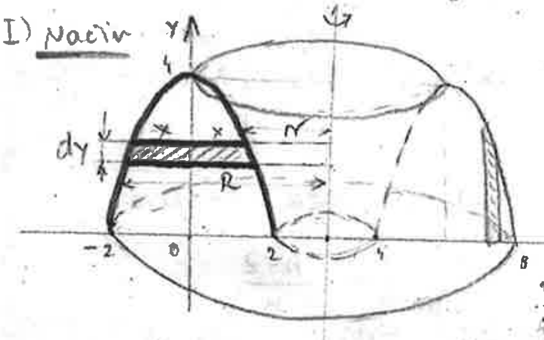
$dV = R^2\pi\, dy - r^2\pi\, dy = \pi\left[(3+x)^2 - (3-x)^2\right] dy = 12\pi x\, dy$

$V = \int_{-2}^{2} 12\pi x\, dy = 24\pi\int_0^2\sqrt{4-y^2}\, dy = 24\pi\left(\frac{y}{2}\sqrt{4-y^2} + 2\sin^{-1}\frac{y}{2}\right)$

$= 24\pi\left(0 + 2\frac{\pi}{2} - 0\right) = \underline{\underline{24\pi^2}}$

4) Naoti volumen tijela koje nastaje rotacijom plohe omeđene parabolom $y = 4 - x^2$ i x-osi oko pravca $x = 3$

I) Način



volumen prstena $dV = R^2\pi\, dy - r^2\pi\, dy \qquad R = 3 + x \quad r = 3 - x$

$dV = \pi\left[(3+x)^2 - (3-x)^2\right] dy = 12\pi x\, dy$

iz $y = 4 - x^2 \Rightarrow x = \sqrt{4-y}$

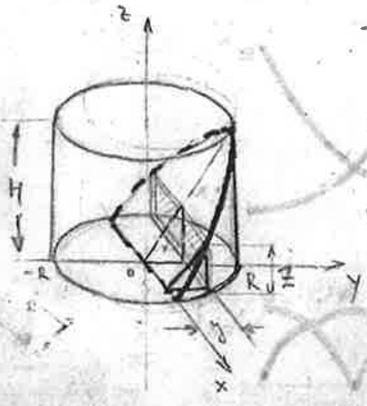
$V = 12\pi\int_0^4 \sqrt{4-y}\, dy = -8\pi(4-y)^{3/2}\Big|_0^4 = \underline{\underline{64\pi}}$

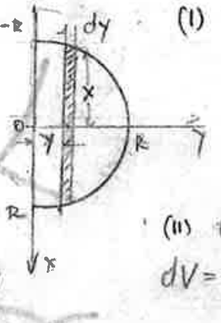
II NAČIN $\qquad dV = 2\pi(3-x)\, y\, dx = 2\pi(3-x)(4-x^2)\, dx$

$V = 2\pi\int_{-2}^{2}(3-x)(4-x^2)\, dx = 2\pi\int_{-2}^{2}(12 - 4x - 3x^2 + x^3)\, dx$

$= 2\pi\left[12x - 2x^2 - x^3 + \frac{x^4}{4}\right]_{-2}^{2} = 2\pi(24 - 8 - 8 + 4 + 24 + 8 - 8 - 4) = 2\pi\cdot 32 = \underline{\underline{64\pi}}$

5) Odredi volumen odsječka uspravnog valjka polumjera baze R i visine H





(I) pravokutni presjek $\quad dV = 2x\cdot z\, dy \qquad z$ iz sličnih trokuta

$\frac{z}{y} = \frac{H}{R} \Rightarrow z = \frac{H}{R}y \qquad$ iz $x^2 + y^2 = R^2 \quad x = \sqrt{R^2 - y^2}$

$V = \frac{2H}{R}\int_0^R y\sqrt{R^2-y^2}\, dy = -\frac{2H}{3R}(R^2 - y^2)^{3/2}\Big|_0^R = \underline{\underline{\frac{2}{3}R^2 H}}$

(II) trokutni presjek

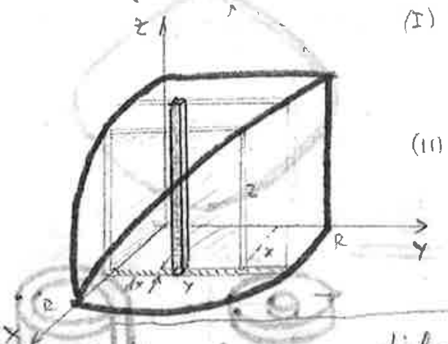
$dV = \frac{y\cdot z}{2}\, dx = \frac{H}{2R}y^2\, dx = \frac{H}{2R}(R^2 - x^2)\, dx$

$V = \frac{H}{2R}\int_{-R}^{R} y^2\, dx = \frac{H}{R}\int_0^R (R^2 - x^2)\, dx = \frac{H}{R}\left[R^2 x - \frac{x^3}{3}\right]_0^R = HR^2\left(1 - \frac{1}{3}\right)$

$V = \underline{\frac{2}{3}R^2 H}$

(6) Naći volumen omeđen sa dva uspravna kružna valjka

$x^2+y^2=R^2$ i $x^2+z^2=R^2$



(I) Način $dV = y\cdot z\,dx = (R^2-x^2)\,dx$ zbog $y=\sqrt{R^2-x^2}$, $z=\sqrt{R^2-x^2}$

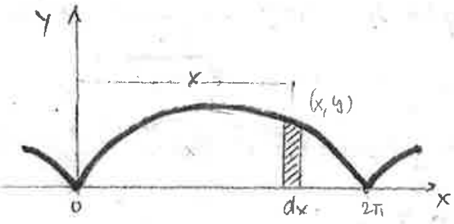
$$V = 8\int_0^R (R^2-x^2)\,dx = 8\left[R^2x-\frac{x^3}{3}\right]_0^R = 8R^3\left(1-\frac{1}{3}\right) = \frac{16}{3}R^3$$

(II) Način pomoću dvostrukog integrala:

$$\frac{V}{8} = \int_0^R\left(\int_{y=0}^{y=\sqrt{R^2-x^2}} z\,dy\right)dx = \int_0^R\left(zy\Big|_{y=0}^{y=\sqrt{R^2-x^2}}\right)dx = \int_0^R (R^2-x^2)\,dx = \frac{2}{3}R^3$$

$$V = 8\cdot\frac{2}{3}R^3 \Rightarrow V = \frac{16}{3}R^3$$

(7) Naći volumen tijela koje se dobiva rotacijom oko y-osi površine između prvog luka cikloide i x-osi



$x = \varphi - \sin\varphi$ $\quad dV = 2\pi x\cdot y\,dx$

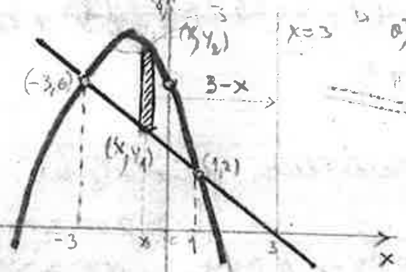
$y = 1-\cos\varphi$ $\quad dx = d\varphi - \cos\varphi\,d\varphi = (1-\cos\varphi)\,d\varphi$

$$V = 2\pi\int_0^{2\pi}(\varphi-\sin\varphi)(1-\cos\varphi)^2\,d\varphi = 2\pi\int_0^{2\pi}(\varphi-\sin\varphi)(1-2\cos\varphi+\cos^2\varphi)$$

$$= 2\pi\int_0^{2\pi}(\varphi - 2\varphi\cos\varphi + \varphi\cos^2\varphi - \sin\varphi + 2\sin\varphi\cos\varphi - \cos^2\varphi\sin\varphi)\,d\varphi$$

$$V = 2\pi\left[\frac{3}{4}\varphi^2 - 2(\varphi\sin\varphi+\cos\varphi) + \frac{1}{2}\left(\frac{1}{2}\varphi\sin 2\varphi + \frac{1}{4}\cos 2\varphi\right) + \cos\varphi + \sin^2\varphi + \frac{1}{3}\cos^3\varphi\right]_0^{2\pi} = 6\pi^3$$

(8) Naći volumen koji se dobiva kad površina omeđena s $y=-x^2-3x+6$ i $y=3-x$ rotira oko a) $x=3$; b) $y=0$



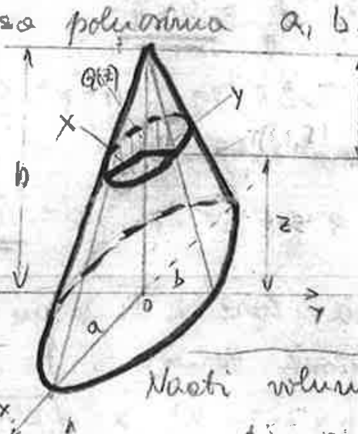
a) $dV = 2\pi(3-x)(y_2-y_1)\,dx$

$$V = 2\pi\int_{-3}^{1}(y_2-y_1)(3-x)\,dx = 2\pi\int_{-3}^{1}(-x^2-2x+3)(3-x)\,dx$$

$$= 2\pi\int_{-3}^{1}(x^3-x^2-9x+9)\,dx = \frac{256}{3}\pi$$

b) $dV = (y_2^2\pi - y_1^2\pi)\,dx$ $\quad V = \pi\int_{-3}^{1}(y_2^2-y_1^2)\,dx = \pi\int_{-3}^{1}(x^4+6x^3-4x^2-30x+27)\,dx = \frac{1792}{15}\pi$

(9) Naći volumen uspravnog stošca visine h čija je baza elipsa sa poluosima a, b.



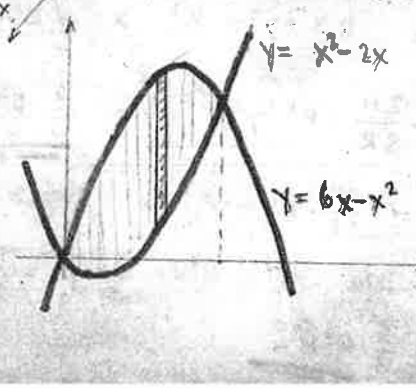
$dV = xy\pi\,dz$ $\quad \frac{x}{a} = \frac{h-z}{h}$ ili $\frac{y}{b} = \frac{h-z}{h}$

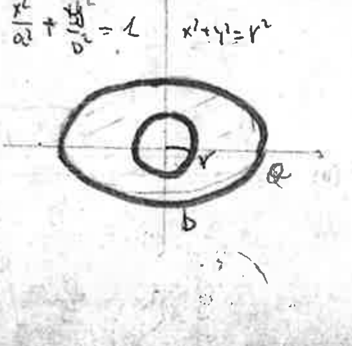
$dV = ab\pi\left(\frac{h-z}{h}\right)^2 dz$ $\quad M(V) = \int_0^h Q(z)\,dz$ $\quad Q(z) = ab\pi\left(\frac{h-z}{h}\right)^2$

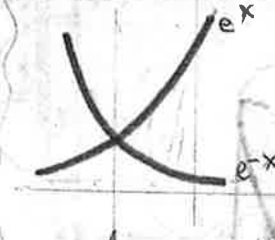
$$V = \frac{ab\pi}{h^2}\int_0^h (h-z)^2\,dz = -\frac{ab\pi}{h^2}\,\frac{(h-z)^3}{3}\Big|_0^h = \frac{1}{3}\pi abh$$

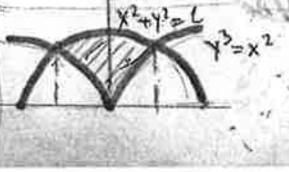
2° $a=b=R \Rightarrow V_{stošca} = \frac{1}{3}R^2\pi\cdot h$

VJEŽBE: Naći volumen:

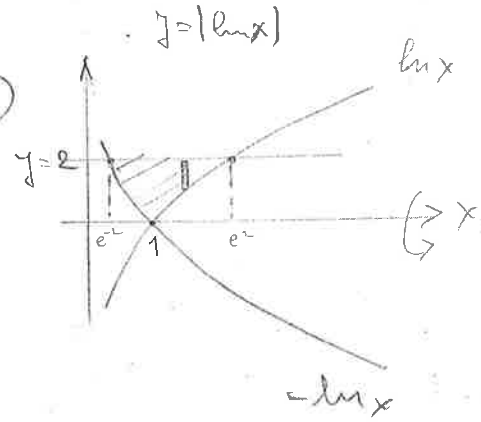








(1)



$y=-\ln x = 2 \Rightarrow \ln x = -2 \quad x = e^{-2}$

$y = \ln x = 2 \Rightarrow \ln x = 2 \quad x = e^{2}$

$$V_x = 2^2\pi\,(e^2 - e^{-2}) - \left[\pi \int_{e^{-2}}^{1} (-\ln x)^2\,dx + \pi \int_{1}^{e^2} (\ln x)^2\,dx\right]$$

$$= \pi\left[4(e^2 - e^{-2}) - \int_{e^{-2}}^{e^2} (\ln x)^2\,dx\right]$$

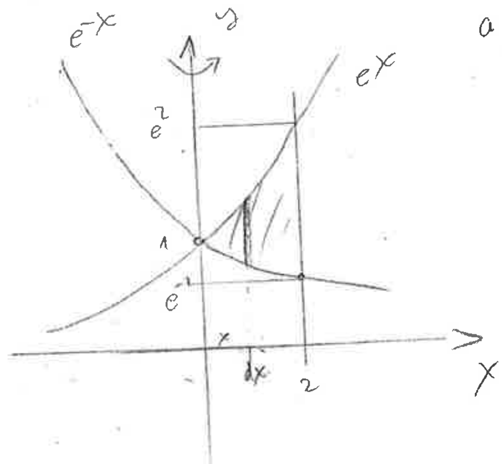
$$= \pi\left[4(e^2 - e^{-2}) - \left(x(\ln x)^2 - 2x\ln x + 2x\right)\Big|_{e^{-2}}^{e^2}\right] =$$

$$= \pi\left[4e^2 - \cancel{4e^{-2}} - \cancel{4e^2} + \cancel{4e^2} - 2e^2 + \cancel{4e^{-2}} + 4e^{-2} + 2e^{-2}\right]$$

$$= \pi\,(2e^2 + 6e^{-2}) = 2\pi\left(e^2 + \frac{3}{e^2}\right)$$

---

(2)



a) oko y-osi

$$dV_y = 2\pi x\,(e^x - e^{-x})\,dx$$

$$V_y = 2\pi \int_0^2 (x e^x - x e^{-x})\,dx =$$

$$= 2\pi\left(e^x(x-1) + e^{-x}(x+1)\Big|_0^2\right)$$

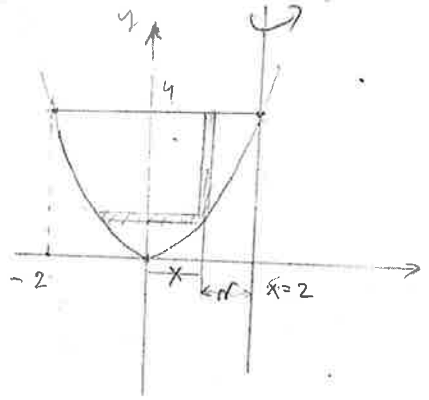
$$= 2\pi\,(e^2 + 3e^{-2})$$

---

b) oko x-osi $\quad dV_x = \pi\,(e^{2x} - e^{-2x})\,dx \quad V_x = \pi\int_0^2 (e^{2x} - e^{-2x})\,dx$

$$= \pi\left(\frac{e^{2x}}{2} + \frac{e^{-2x}}{2}\right)\Big|_0^2 = \pi\left(\frac{e^4}{2} + \frac{e^{-4}}{2} - 1\right) = \pi\,(\mathrm{ch}\,4 - 1)$$

1)



$y=x^2 \quad y=4 \quad$ oko $\quad x=2$

(I) $dV = 2\pi r\,(4-y)\,dx = 2\pi(2-x)(4-y)\,dx$

$$V = 2\pi\int_{-2}^{2}(2-x)(4-x^2)\,dx = 8\pi\int_{0}^{2}(4-x^2)\,dx$$

$$= 8\pi\left(4x-\frac{x^3}{3}\right)\Big|_0^2 = 8\pi\left(8-\frac{8}{3}\right) = \frac{128\pi}{3}$$

(II) uz $x=\sqrt{y}$: $\quad dV = \left[(2+x)^2\pi - (2-x)^2\pi\right]dy$

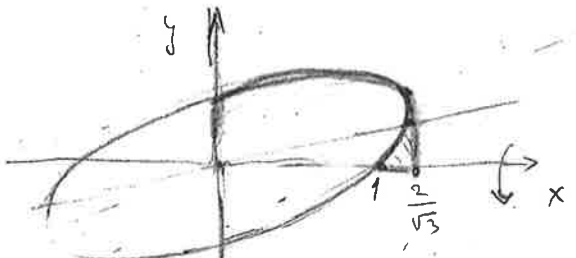
$dV = 8\pi x\,dy \quad \Rightarrow \quad V = 8\pi\int_0^4 \sqrt{y}\,dy = 8\pi\cdot\frac{2}{3}y^{\frac{3}{2}}\Big|_0^4 = \frac{16\pi}{3}\cdot 8 = \frac{128\pi}{3}$

ili zbog $dy = 2x\,dx$
$y=x^2$: $\quad V = 8\pi\int_0^2 x\cdot 2x\,dx = 16\pi\,\frac{x^3}{3}\Big|_0^2 = \frac{128\pi}{3}$

Volumen

① Krivulja $x^2 - xy + y^2 = 1$ rotira oko x-osi

$$y = \frac{x}{2} \pm \sqrt{1 - \frac{3}{4}x^2} \qquad \left(V_x = \frac{8\pi}{3} \right)$$



$$V_x = 2\pi \left[\int_0^{\frac{2}{\sqrt{3}}} \left(\frac{x}{2} + \sqrt{1 - \frac{3}{4}x^2} \right)^2 dx - \int_1^{\frac{2}{\sqrt{3}}} \left(\frac{x}{2} - \sqrt{1 - \frac{3}{4}x^2} \right)^2 dx \right]$$

$$= 2\pi \left[\int_0^{\frac{2}{\sqrt{3}}} \left(1 - \frac{x^2}{2} + x\sqrt{1 - \frac{3}{4}x^2} \right) dx - \int_1^{\frac{2}{\sqrt{3}}} \left(1 - \frac{x^2}{2} - x\sqrt{1 - \frac{3}{4}x^2} \right) dx \right]$$

$$= 2\pi \left[\left(x - \frac{x^3}{6} \right) \Big|_0^1 - \frac{4}{9} \left(1 - \frac{3}{4}x^2 \right)^{\frac{3}{2}} \Big|_0^{\frac{2}{\sqrt{3}}} - \frac{4}{9} \left(1 - \frac{3}{4}x^2 \right)^{\frac{3}{2}} \Big|_1^{\frac{2}{\sqrt{3}}} \right] =$$

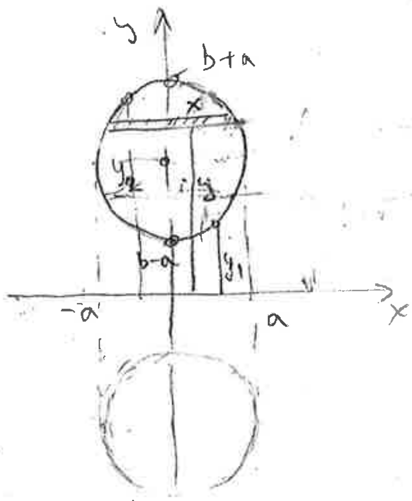
$$= 2\pi \left[\left(1 - \frac{1}{6} \right) + \frac{4}{9} + \frac{4}{9} \left(1 - \frac{3}{4} \right)^{\frac{3}{2}} \right] = 2\pi \left(\frac{5}{6} + \frac{4}{9} + \frac{4}{9} \cdot \frac{1}{8} \right)$$

$$= 2\pi \left(\frac{15 + 8 + 1}{18} \right) = \frac{\pi \cdot 24}{9} = \frac{8\pi}{3}$$

TORUS

② $x^2 + (y - b)^2 = a^2 \qquad 0 < a \leq b \qquad y_2 = b + \sqrt{a^2 - x^2}$

$x = \sqrt{a^2 - (y-b)^2} \qquad y_1 = b - \sqrt{a^2 - x^2}$



$$V_x = \pi \int_{-a}^{a} (y_2^2 - y_1^2)\, dx = \pi \int_{-a}^{a} (y_2 - y_1)(y_2 + y_1)\, dx$$

$$= 2\pi \int_0^a 2\sqrt{a^2 - x^2} \cdot 2b = 8\pi b \int_0^a \sqrt{a^2 - x^2}\, dx$$

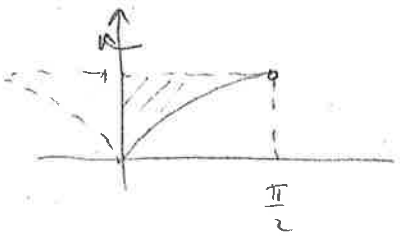
$$= 8\pi a^2 b \int_0^{\frac{\pi}{2}} \cos^2 t\, dt = 2\pi^2 a^2 b$$

II) $y - b = a \sin t$

$$V = \int_{b-a}^{b+a} 2y\pi \cdot 2x\, dy = 4\pi \int_{b-a}^{b+a} y\sqrt{a^2 - (y-b)^2}\, dy = 4\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (b + a \sin t)\, a^2 \cos^2 t\, dt$$

$$= 8\pi b a^2 \int_0^{\frac{\pi}{2}} \cos^2 t\, dt = 8\pi a^2 b \cdot \frac{\pi}{4} = 2\pi^2 a^2 b$$

Volumen

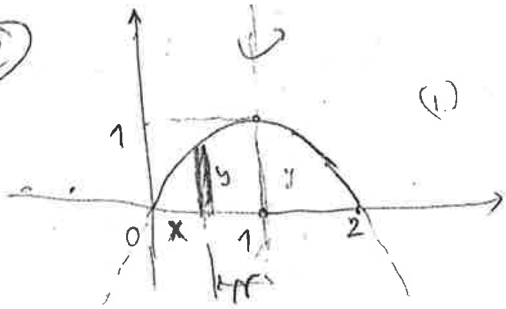


$$V_y = \pi \int_0^1 x^2\, dy = \pi \int_0^1 (\arcsin y)^2\, dy$$

$$= \pi \int_0^{\pi/2} t^2 \cos t\, dt \quad - - - \quad = \frac{\pi(\pi^2 - 8)}{4}$$

---

(2)



$$y = 2x - x^2 \qquad \text{oko} \qquad x = 1$$

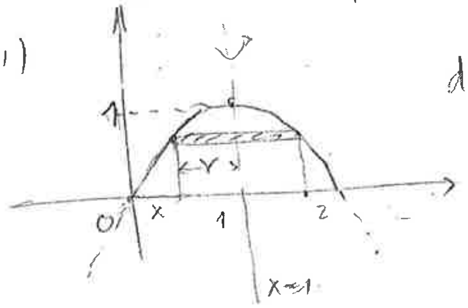
i) $$dV_1 = 2\pi r y\, dx \qquad r = 1 - x$$

$$V_1 = 2\pi \int_0^1 (1-x)(2x - x^2)\, dx =$$

$$= 2\pi \int_0^1 (2x - x^2 - 2x^2 + x^3)\, dx = 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\, dx =$$

$$= 2\pi \left(\frac{x^4}{4} - x^3 + x^2 \right)\Big|_0^1 = 2\pi \left(\frac{1}{4} - 1 + 1 \right) = \frac{\pi}{2}$$

ii)



$$dV_1 = r^2 \pi\, dy \qquad r = 1 - x \qquad x^2 - 2x + y = 0$$

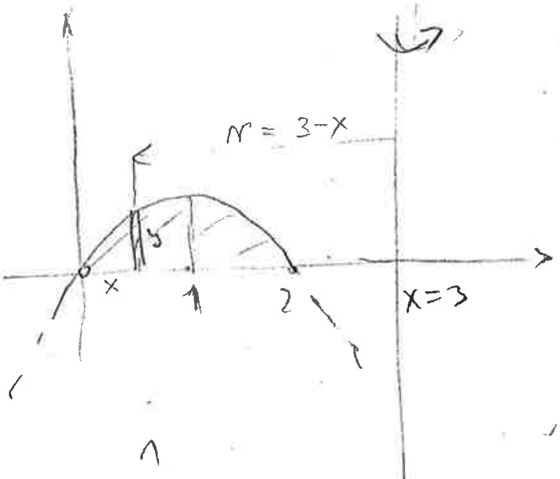
$$x = 1 - \sqrt{1-y}$$

$$V_1 = \pi \int_0^1 (1-y)\, dy = \pi \left(y - \frac{y^2}{2} \right)\Big|_0^1 \qquad r = \sqrt{1-y}$$

$$V_1 = \pi \left(1 - \frac{1}{2} \right) = \frac{\pi}{2}$$

---

(3)



$$dV_3 = 2r\pi \cdot y\, dx = 2\pi(3-x)\, y\, dx$$

$$V_3 = 2\pi \int_0^2 (3-x)(2x - x^2)\, dx$$

$(3-x)x(2-x)$ pazi!

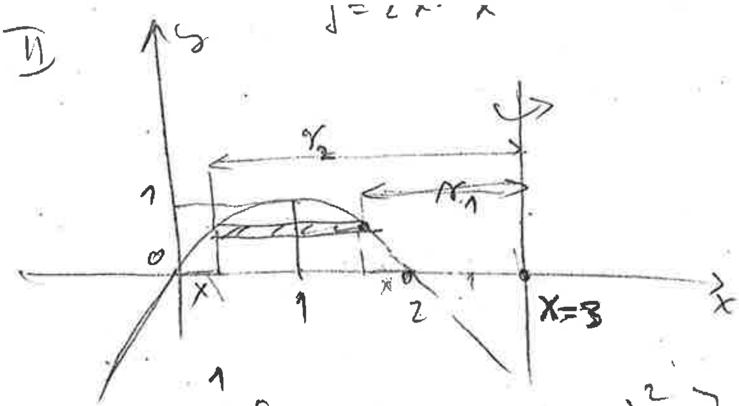
$x \ldots x + 1$ (nadomestimo! P.

$$= 2\pi \int_{-1}^{1} (3 - x - 1)(x+1)(2 - x - 1)\, dx$$

$$= 2\pi \int_{-1}^{1} (2-x)(1+x)(1-x)\, dx = 2\pi \int_{-1}^{1} (2-x)(1-x^2)\, dx = 4\pi \int_0^1 2(1-x^2)\, dx =$$

$$= 8\pi \left(x - \frac{x^3}{3} \right)\Big|_0^1 = 8\pi \left(1 - \frac{1}{3} \right) = \frac{16\pi}{3}$$

(3) II



$$dV_3 = (r_2^2\pi - r_1^2\pi)\,dy$$

$$r_2 = 3 - x \qquad x = 1 - \sqrt{1-y}$$

$$r_1 = 1 + x$$

$$V_3 = \pi\int_0^1 \left[(3-x)^2 - (1+x)^2\right] dy = \pi\int_0^1 4\,(2-2x)\,dy =$$

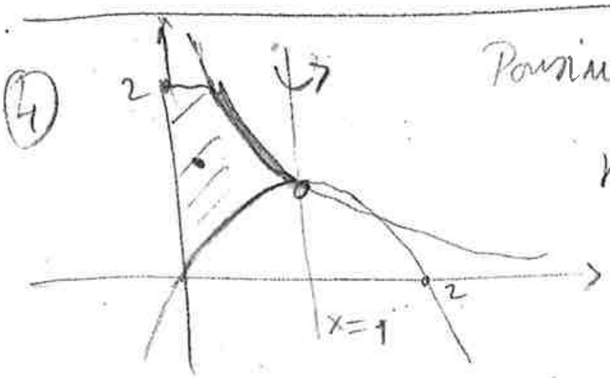
$$= 8\pi\int_0^1 (1-x)\,dy = 8\pi\int_0^1 \sqrt{1-y}\,dy = 8\pi\left(-\frac{2}{3}\right)(1-y)^{\frac{3}{2}}\Big|_0^1 = \frac{16\pi}{3}$$

ili zbog $y = 2x - x^2 \Rightarrow dy = (2-2x)\,dx$!

$$8\pi \cdot 2\int_0^1 (1-x)^2\,dx = 16\pi\,\frac{(1-x)^3}{-3}\Big|_0^1 = \frac{16\pi}{3}$$

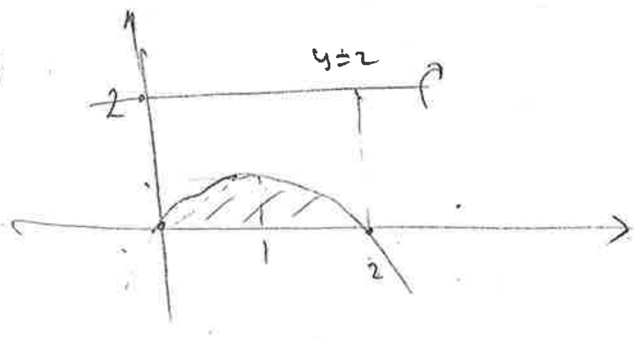
---

(4)

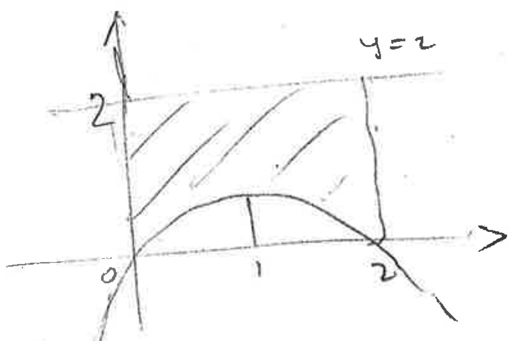


Površina omeđena sa $y = 2x - x^2$, $y = \frac{1}{x}$, $x = 0$, $y = 2$ rotira oko $x = 1$

(5)



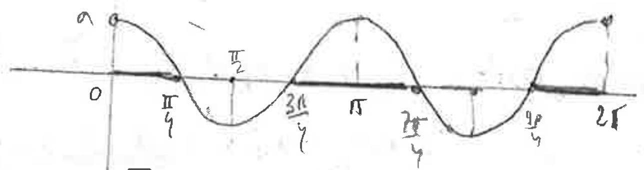
$y = 2x - x^2$
$y = 0$

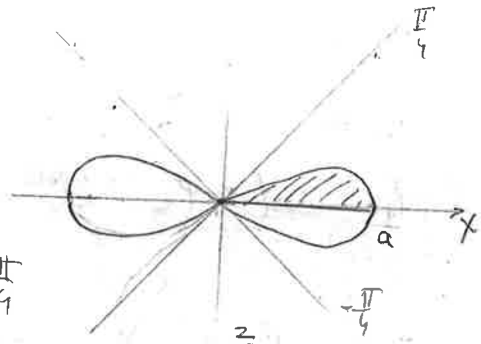


$y = 2x - x^2$, $y = 2$
$x = 0$, $x = 2$

② Volumen nastao rotacijom $r^2 = a^2\cos 2\varphi$ oko

a) x-osi   b) y-osi





a)

$$V_x = \frac{4\pi}{3}\int_0^{\frac{\pi}{4}} a^3\cos^{\frac32} 2\varphi\,\sin\varphi\,d\varphi = \frac{4\pi a^3}{3}\int_0^{\frac{\pi}{4}}(2\cos^2\varphi-1)^{\frac32}\sin\varphi\,d\varphi$$

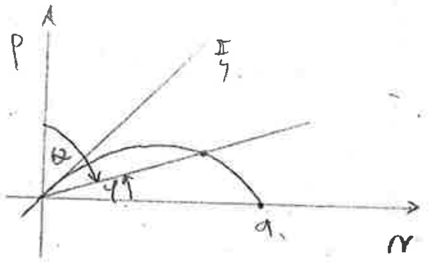
smj. $\sqrt2\cos\varphi = t$ $\varphi = 0$ $t=\sqrt2$ $-\sin\varphi\,d\varphi = \frac{dt}{\sqrt2}$

$-\sqrt2\sin\varphi\,d\varphi = dt$ $\varphi=\frac{\pi}{4}$ $t=1$

$$V_x = \frac{4\pi a^3}{3\sqrt2}\int_1^{\sqrt2}(t^2-1)^{\frac32}dt = \frac{4\pi a^3}{3\sqrt2}\cdot\frac14\left[t\,(t^2-1)^{\frac32} - \frac32 t\sqrt{t^2-1} + \frac32\ln\left(t+\sqrt{t^2-1}\right)\right]_1^{\sqrt{}}$$

$$V_x = \frac{\pi a^3}{3\sqrt2}\left[\sqrt2 - \frac32\sqrt2 + \frac32\ln(1+\sqrt2)\right] = \frac{\pi a^3}{4}\left[\sqrt2\ln(1+\sqrt2) - \frac23\right]$$

b)



Rotacija oko ρ-osi (y-os)

$$V_y = \frac23\pi\int_{\theta_1}^{\theta_2}\rho^3(\theta)\sin\theta\,d\theta = \frac{4\pi}{3}\int_{-\frac{\pi}{4}}^{-\frac{\pi}{2}}\rho^3(\theta)\sin\theta\,d\theta$$

Vidimo da je $\varphi-\theta = \frac{\pi}{2}$ i $\rho(\theta) = r(\varphi)$

$\varphi = \theta + \frac{\pi}{2}$ $\theta = -\frac{\pi}{4} \Rightarrow \varphi = \frac{\pi}{4}$ $\sin\theta = \sin\left(\varphi - \frac{\pi}{2}\right) = -\cos\varphi$

$\theta = -\frac{\pi}{2} \Rightarrow \varphi = 0$ $d\theta = d\varphi$!

$$V_y = \frac{4\pi a^3}{3}\int_0^{\frac{\pi}{4}}(\cos 2\varphi)^{\frac32}\cos\varphi\,d\varphi = \frac{4\pi a^3}{3}\int_0^{\frac{\pi}{4}}(1-2\sin^2\varphi)^{\frac32}\cos\varphi\,d\varphi$$

$\sqrt2\sin^2\varphi = z$ $\varphi = 0 \Rightarrow z = 0$ $\sqrt2\cos\varphi\,d\varphi = dz$

$\varphi = \frac{\pi}{4} \Rightarrow z = 1$

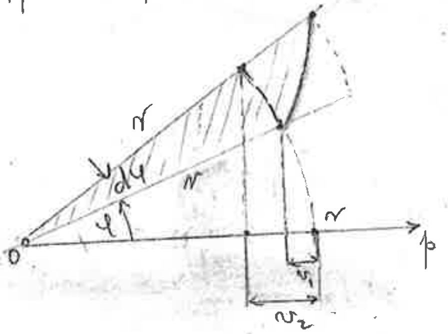
$$\frac{4\pi a^3}{3\sqrt2}\int_0^1(1-z^2)^{\frac32}dz =$$

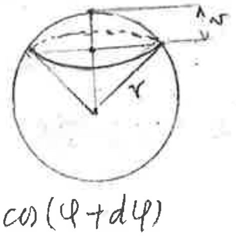
ponovo smj: $z = \sin t$ $dz = \cos t\,dt$ $z=0$ $t=0$ $z=1$ $t=\frac{\pi}{2}$

$$= \frac{4\pi a^3}{3\sqrt2}\int_0^{\frac{\pi}{2}}\cos^4 t\,dt = \frac{4\pi a^3}{3\sqrt2}\cdot\frac38\,\frac{\pi}{2} \Rightarrow V_y = \frac{\pi^2 a^3}{4\sqrt2}$$

Volumen rotacionog tijela u polarnim koordinatama

Izvedite formulu za volumen tijela nastalog rotacijom sektora ograničenog sa $r=r(\varphi)$, $\varphi=\varphi_1$ i $\varphi=\varphi_2$ oko polarne osi $(0\le\varphi_1<\varphi_2\le\pi)$





Volumen kuglinog isječka

$$V=\frac{2}{3}r^2\pi\cdot v$$

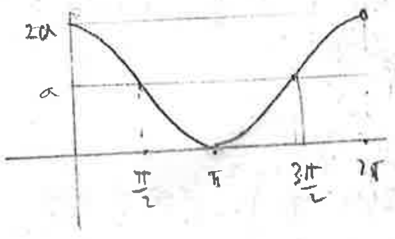
$$v_2=r-r\cos(\varphi+d\varphi)$$
$$v_1=r-r\cos\varphi$$

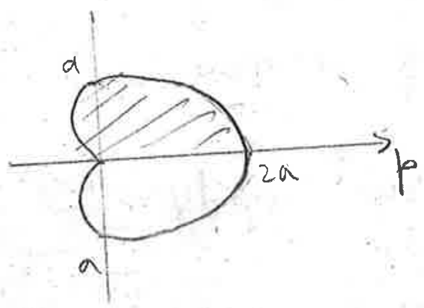
$$dV'=\frac{2}{3}r^2\pi v_2-\frac{2}{3}r^2\pi v_1=\frac{2}{3}r^2\pi(v_2-v_1)=$$
$$=\frac{2}{3}r^2\pi[\cancel{r}-r\cos(\varphi+d\varphi)-\cancel{r}+r\cos\varphi]=\frac{2}{3}r^3\pi[\cos\varphi-\cos(\varphi+d\varphi)]$$
$$=\frac{2}{3}r^3\pi\cdot 2\sin\frac{2\varphi+d\varphi}{2}\cdot\sin\frac{d\varphi}{2}\approx\frac{2}{3}r^3\pi\cdot 2\sin\varphi\,\frac{d\varphi}{2}$$

$$\Rightarrow dV=\frac{2}{3}r^3\pi\sin\varphi\,d\varphi\Rightarrow V=\frac{2\pi}{3}\int_{\varphi_1}^{\varphi_2}r^3(\varphi)\sin\varphi\,d\varphi$$

---

① Izračunaj volumen tijela nastalog rotacijom $r=a(1+\cos\varphi)$ $(0<\varphi\le 2\pi)$ oko polarne osi





$$V=\frac{2\pi}{3}\int_0^\pi r^3\sin\varphi\,d\varphi=\frac{2\pi}{3}\int_0^\pi a^3(1+\cos\varphi)^3\sin\varphi\,d\varphi$$

supst. $1+\cos\varphi=t$ $\quad -\sin\varphi\,d\varphi=dt$

$\varphi=0 \quad t=2$
$\varphi=\pi \quad t=0$

$$\frac{2\pi}{3}\int_0^2 a^3t^3\,dt=\frac{2\pi a^3}{3}\,\frac{t^4}{4}\Big|_0^2$$

$$V=\frac{8}{3}a^3\pi$$