9 Sestr. díl.

jed. a konst kód.

Linearni sistem I-og reda s konstantnim koeficijentima

$$\dot{x}_i(t) = \sum_{k=1}^{n} a_{ik} x_k(t) + f_i(t) \qquad i = 1, 2, \dots n \qquad f_i = 0 \Rightarrow \text{Homogen sistem}$$

$$x = (x_1, x_2 \cdots x_n) \qquad f(t) = (f_1, f_2 \cdots f_n) \qquad A = [a_{ik}]$$

$$\begin{cases} \dot{x}(t) = A\,x(t) + f(t) \\ x(t_0) = x_0 \end{cases} \quad \text{Cauchy-ev problem} \qquad \text{Homogen sistem} \begin{cases} \dot{x}(t) = A \cdot x(t) \\ x(t_0) = x_0 \end{cases}$$

(I) Svođenje na dif. jednačinu n-tog reda

$$\dot{x}_1(t) = x_2(t) + 1 \Rightarrow x_2 = \dot{x}_1 - 1 \Rightarrow \dot{x}_2 = \ddot{x}_1$$

$$\dot{x}_2(t) = x_1(t) + 1 \qquad \text{sada je } \ddot{x}_1 = x_1 + 1 \quad \text{tj.}$$

$$\ddot{x}_1 - x_1 = 1 \quad \text{dif. jed 2-og reda s konstantnim koef.}$$

$$r^2 - 1 = 0 \quad r_{1,2} = \pm 1 \Rightarrow x_1 = c_1 e^{t} + c_2 e^{-t} - 1$$

dok je $x_2 = \dot{x}_1 - 1 = c_1 e^{t} - c_2 e^{-t} - 1$

konačno:

$$x_1 = c_1 e^{t} + c_2 e^{-t} - 1$$
$$x_2 = c_1 e^{t} - c_2 e^{-t} - 1$$

(II) Homogen sistem: Matrično rješavanje pomoću matrice e^{At}

Def. $e^{At} = I + \frac{1}{1!} A t + \frac{1}{2!} A^2 t^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} A^n t^n$

TEOREM: $e^{At} = \alpha_{n-1} A^{n-1} t^{n-1} + \alpha_{n-2} A^{n-2} \cdot t^{n-2} + \cdots + \alpha_2 A^2 t^2 + \alpha_1 A t + \alpha_0 I$

za $\boxed{n = 2 \quad e^{At} = \alpha_1 A t + \alpha_0 I}$

$n = 3 \quad e^{At} = \alpha_2 A^2 t^2 + \alpha_1 A t + \alpha_0 I$

TEOREM: $n = 2$

$$\boxed{\begin{aligned} e^{\lambda_1 t} &= \alpha_1 \lambda_1 t + \alpha_0 \\ e^{\lambda_2 t} &= \alpha_1 \lambda_2 t + \alpha_0 \end{aligned}} \qquad \lambda_1 \neq \lambda_2 \qquad (*)$$

$\lambda_1 = \lambda_2$

$$e^{\lambda_1 t} = \alpha_1 \lambda_1 t + \alpha_0$$
$$e^{\lambda_1 t} = \alpha_1$$

λ_1 i λ_2 su svojstvene vrijednosti za A

$|A - \lambda I| = 0 \Rightarrow \lambda_1$ i λ_2

PRIMJER $\dot{x}_1(t) = x_2(t)$
$\dot{x}_2(t) = x_1(t)$ $\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \dot{x} = Ax$

Rješenje sistema $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = e^{At} x(0)$

Naćimo prvo sopstvene vrijednosti matrice $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \quad \lambda_{1,2} = \pm 1$

$e^{At} = \alpha_1 At + \alpha_0 I = \alpha_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} t + \alpha_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \alpha_1 t \\ \alpha_1 t & 0 \end{bmatrix} + \begin{bmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{bmatrix}$

$e^{At} = \begin{bmatrix} \alpha_0 & \alpha_1 t \\ \alpha_1 t & \alpha_0 \end{bmatrix}$ Sada iz sistema

$\lambda_1 = 1$
$\lambda_2 = -1$ primjenimo (*)
$e^{t} = \alpha_1 t + \alpha_0$
$e^{-t} = \alpha_1 (-t) + \alpha_0$ Naođimo α_1 i α_0

$2\alpha_1 t = e^{t} - e^{-t} \Rightarrow \alpha_1 t = \dfrac{e^{t} - e^{-t}}{2}$ i

$\alpha_0 = e^{t} - \dfrac{e^{t} - e^{-t}}{2} \Rightarrow \alpha_0 = \dfrac{e^{t} + e^{-t}}{2}$ tj.

$e^{At} = \begin{bmatrix} \dfrac{e^{t}+e^{-t}}{2} & \dfrac{e^{t}-e^{-t}}{2} \\ \dfrac{e^{t}-e^{-t}}{2} & \dfrac{e^{t}+e^{-t}}{2} \end{bmatrix}$ sada je uz $x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$

$x = e^{At} x_0 = \dfrac{1}{2} \begin{bmatrix} e^{t}+e^{-t} & e^{t}-e^{-t} \\ e^{t}-e^{-t} & e^{t}+e^{-t} \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} x_{10}e^{t} + x_{10}e^{-t} + x_{20}e^{t} - x_{20}e^{-t} \\ x_{10}e^{t} - x_{10}e^{-t} + x_{20}e^{t} + x_{20}e^{-t} \end{bmatrix}$

tj. $x_1 = (x_{10} + x_{20}) e^{t} + (x_{10} - x_{20}) e^{-t} \Rightarrow x_1 = c_1 e^{t} + c_2 e^{-t}$
$x_2 = (x_{10} + x_{20}) e^{t} - (x_{10} - x_{20}) e^{-t}$ $\quad x_2 = c_1 e^{t} - c_2 e^{-t}$

INAČE: Svođenjem na jednačinu II-og reda $\Rightarrow$

$\ddot{x}_1 = \dot{x}_2 = x_1 \Rightarrow \ddot{x}_1 - x_1 = 0 \quad r^2 - 1 = 0 \quad r_{1,2} = \pm 1$

$x_1 = c_1 e^{t} + c_2 e^{-t}$ zbog $x_2 = \dot{x}_1 \Rightarrow$

$x_2 = c_1 e^{t} - c_2 e^{-t}$

① $\dot{x}_1 = -7x_1 + x_2 \Longrightarrow x_2 = \dot{x}_1 + 7x_1 \Rightarrow \dot{x}_2 = \ddot{x}_1 + 7\dot{x}_1$

$\dot{x}_2 = -2x_1 - 5x_2$ (2) $\quad$ inlocuim în (2) dobim

$$\ddot{x}_1 + 7\dot{x}_1 = -2x_1 - 5(\dot{x}_1 + 7x_1) = -2x_1 - 5\dot{x}_1 - 35x_1$$

$$\ddot{x}_1 + 12\dot{x}_1 + 37x_1 = 0$$

$$r^2 + 12r + 37 = 0 \Rightarrow r_{1,2} = -6 \pm i$$

$$x_1 = e^{-6t}(c_1 \sin t + c_2 \cos t)$$

$$x_2 = \dot{x}_1 + 7x_1 = e^{-6t}[(c_1 + c_2)\cos t + (c_2 - c_1)\sin t]$$

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① $\dot{x}_1 = x_2$ $\qquad \ddot{x}_2 = \dot{x}_1 + e^t = x_2 + e^t$

$\dot{x}_2 = x_1 + e^t$ $\qquad \ddot{x}_2 - x_2 = e^t \qquad r^2 - 1 = 0 \quad r_{1,2} = \pm 1$

$x_{2p} = kte^t \quad \dot{x}_{2p} = ke^t(1+t) \quad \ddot{x}_{2p} = ke^t(2+t)$

$k(2 + \cancel{t} - \cancel{t}) = 1 \qquad k = \frac{1}{2} \qquad x_2 = c_1 e^t + c_2 e^{-t} + \frac{t}{2} e^t$

$x_1 = \dot{x}_2 - e^t = c_1 e^t - c_2 e^{-t} + \frac{1}{2} e^t + \frac{t}{2} e^t - e^t$

$x_1 = c_1 e^t - c_2 e^{-t} + \frac{1}{2}(t-1) e^t$

Riješi sistem:
$$\dot x_1 = x_1 + 2x_2 \quad (1)$$
$$\dot x_2 = 4x_1 + 3x_2 \quad (2)$$
iz 1) deriviramo

I) $x_2 = \frac{1}{2}(\dot x_1 - x_1) \Rightarrow \dot x_2 = \frac{1}{2}(\ddot x_1 - \dot x_1)$ uvrstimo u (2)

$$\frac{1}{2}(\ddot x_1 - \dot x_1) = 4x_1 + \frac{3}{2}(\dot x_1 - x_1) \quad / \cdot 2 \Rightarrow$$
$$\ddot x_1 - \dot x_1 = 8x_1 + 3\dot x_1 - 3x_1 \Rightarrow$$
$$\ddot x_1 - 4\dot x_1 - 5x_1 = 0 \Rightarrow r^2 - 4r - 5 = 0 \quad (r+1)(r-5) = 0$$
$$r_1 = -1, \quad r_2 = 5$$
$$x_1 = c_1 e^{-t} + c_2 e^{5t}$$
$$x_2 = \frac{1}{2}(\dot x_1 - x_1) = \frac{1}{2}(-c_1 e^{-t} + 5c_2 e^{5t} - c_1 e^{-t} - c_2 e^{5t})$$
$$x_2 = -c_1 e^{-t} + 2c_2 e^{5t}$$

II) Matrično pomoću e^{At} tj. $X = e^{At}K$

$$\begin{bmatrix}\dot x_1\\ \dot x_2\end{bmatrix} = \begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \Rightarrow A = \begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix} \quad |A - \lambda I| = 0$$

$$\begin{vmatrix}1-\lambda & 2\\ 4 & 3-\lambda\end{vmatrix} = 0 \Rightarrow (1-\lambda)(3-\lambda) - 8 = 0 \quad 3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$$
$$\lambda^2 - 4\lambda - 5 = 0 \Rightarrow \lambda_1 = -1, \quad \lambda_2 = 5$$

$$e^{At} = \alpha_1 At + \alpha_0 I = \begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}\alpha_1 t + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\alpha_0 \Rightarrow e^{At} = \begin{bmatrix}\alpha_1 t + \alpha_0 & 2\alpha_1 t\\ 4\alpha_1 t & 3\alpha_1 t + \alpha_0\end{bmatrix}$$

$$e^{\lambda_1 t} = \alpha_1 \lambda_1 t + \alpha_0 \qquad e^{-t} = -\alpha_1 t + \alpha_0$$
$$e^{\lambda_2 t} = \alpha_1 \lambda_2 t + \alpha_0 \qquad e^{5t} = 5\alpha_1 t + \alpha_0$$
$$\Big\}\ominus \Rightarrow 6\alpha_1 t = e^{5t} - e^{-t}$$
$$\alpha_1 t = \frac{1}{6}(e^{5t} - e^{-t})$$

$$\alpha_0 = e^{-t} + \alpha_1 t = e^{-t} + \frac{1}{6}e^{5t} - \frac{1}{6}e^{-t}$$
$$\alpha_0 = \frac{1}{6}(e^{5t} + 5e^{-t})$$ sada je:

$$X = e^{At}\cdot K = \begin{bmatrix}\frac{1}{3}e^{5t} + \frac{2}{3}e^{-t} & \frac{1}{3}e^{5t} - \frac{1}{3}e^{-t}\\ \frac{2}{3}e^{5t} - \frac{2}{3}e^{-t} & \frac{2}{3}e^{5t} + \frac{1}{3}e^{-t}\end{bmatrix}\begin{bmatrix}K_1\\ K_2\end{bmatrix} \Rightarrow$$

$$x_1 = \frac{K_1}{3}e^{5t} + \frac{2K_1}{3}e^{-t} + \frac{K_2}{3}e^{5t} - \frac{K_2}{3}e^{-t} = \frac{1}{3}(K_1+K_2)e^{5t} + \frac{1}{3}(2K_1 - K_2)e^{-t} = c_1 e^{-t} + c_2 e^{5t}$$
$$x_1 = \frac{2K_1}{3}e^{5t} - \frac{2K_1}{3}e^{-t} + \frac{2}{3}K_2 e^{5t} + \frac{K_2}{3}e^{-t} = \frac{2}{3}(K_1+K_2)e^{5t} - \frac{1}{3}(2K_1 - K_2)e^{-t} = -c_1 e^{-t} + 2c_2 e^{5t}$$

Riješite sistem $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \boxed{\dot{X}(t) = A X(t)}$$

$x_1(0) = 1$

$x_2(0) = -1$

$$X(t) = e^{At} \cdot X(0) \qquad X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \qquad \lambda_{1,2} = \pm i$$

$$e^{At} = \alpha_1 A t + \alpha_0 I = \begin{bmatrix} 0 & \alpha_1 t \\ -\alpha_1 t & 0 \end{bmatrix} + \begin{bmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{bmatrix} = \begin{bmatrix} \alpha_0 & \alpha_1 t \\ -\alpha_1 t & \alpha_0 \end{bmatrix}$$

$$\begin{aligned} e^{\lambda_1 t} &= \alpha_1 \lambda_1 t + \alpha_0 \\ e^{\lambda_2 t} &= \alpha_1 \lambda_2 t + \alpha_0 \end{aligned} \Rightarrow \left.\begin{aligned} e^{-it} &= -\alpha_1 i t + \alpha_0 \\ e^{it} &= \alpha_1 i t + \alpha_0 \end{aligned}\right\} \oplus \quad \alpha_0 = \frac{e^{it} + e^{-it}}{2}$$

$\alpha_0 = \cos t$

$$2\alpha_1 i t = e^{it} - e^{-it} \Rightarrow \alpha_1 t = \frac{e^{it} - e^{-it}}{2i} = \sin t$$

$\alpha_1 t = \sin t$

$$X(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \cos t - \sin t \\ -\sin t - \cos t \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$x_1(t) = \cos t - \sin t$

$x_2(t) = -\cos t - \sin t$

Kontrolirajmo na II način:

$$\dot{x}_1 = x_2 \Rightarrow \ddot{x}_1 = \dot{x}_2 = -x_1 \Rightarrow \ddot{x}_1 + x_1 = 0$$

$$r^2 + 1 = 0 \qquad r_{1,2} = \pm i$$

$x_1(t) = C_1 \sin t + C_2 \cos t$

$x_2(t) = C_1 \cos t - C_2 \sin t$

početni uvjeti $x_1(0) = 1$, $x_2(0) = -1$ $\Rightarrow$

$x_1(0) = 1 = C_2$ $\Rightarrow$ $x_1(t) = -\sin t + \cos t$

$x_2(0) = -1 = C_1$ $\quad$ $x_2(t) = -\cos t - \sin t$

Slučaj višestrukih korijena.

$$\dot{x}_1 = x_1 \qquad x_1(0) = 1$$
$$\dot{x}_2 = x_1 + x_2 \qquad x_2(0) = 0$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = 0 \qquad (1-\lambda)^2 = 0 \qquad \lambda_{1,2} = 1$$

$$e^{At} = \alpha_1 At + \alpha_0 I \quad : \quad e^{t} = \alpha_1 t + \alpha_0$$

$$e^{\lambda_1 t} = \alpha_1 \lambda_1 t + \alpha_0 \qquad e^{t} = \alpha_1 \quad \Rightarrow \alpha_0 = e^t - e^t \cdot t$$

$$e^{\lambda_1 t} = \alpha_1$$

$$\alpha_1 = e^t \ ; \quad \alpha_0 = (1-t)e^t$$

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \alpha_1 t + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \alpha_0 = \begin{bmatrix} \alpha_1 t + \alpha_0 & 0 \\ \alpha_1 t & \alpha_1 t + \alpha_0 \end{bmatrix}$$

$$X(t) = e^{At} x(0) = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t \\ te^t \end{bmatrix} \Rightarrow \begin{array}{l} x_1(t) = e^t \\ x_2(t) = te^t \end{array}$$

Kontrola: iz $\dot{x}_1 = x_1 \Rightarrow \ln x_1 = \ln C + t \Rightarrow x_1 = ce^t$;

$x_1(0) = c = 1 \Rightarrow x_1(t) = e^t \Rightarrow$

$\dot{x}_2 = e^t + x_2 \Rightarrow \dot{x}_2 - x_2 = e^t$ linearna

$$\frac{\dot{x}_2}{x_2} - 1 = \frac{e^t}{x_2} = \frac{\dot{z}}{z} \Rightarrow \ln x_2 - t = \ln z \Rightarrow x_2 = ze^t$$

$$\frac{e^t}{ze^t} = \frac{\dot{z}}{z} \Rightarrow \dot{z} = 1 \Rightarrow z = t + C_1$$

$$x_2 = C_1 e^t + te^t$$

$$x_2(0) = 0 = C_1 \Rightarrow x_2 = te^t$$

$$\dot{x}_1 = x_1 - 2x_2 \qquad x_1 = c_1 e^{3t} - c_2 e^{-t}$$
$$\dot{x}_2 = -2x_1 + x_2 \qquad x_2 = -c_1 e^{3t} + c_2 e^{-t}$$

$$\dot{x}_1 = x_1 + 2x_2 \qquad x_1 = c_1 e^{5t} + c_2 e^{-5t}$$
$$\dot{x}_2 = 12x_1 - x_2 \qquad x_2 = 2c_1 e^{5t} - 3c_2 e^{-5t}$$

$$\dot{x}_1 = -3x_1 - x_2 \qquad x_1 = (c_1 + c_2 t)\, e^{-2t}$$
$$\dot{x}_2 = x_1 - x_2 \qquad x_2 = -(c_1 + c_2 + t c_2)\, e^{-2t}$$

$$\dot{x}_1 = 4x_1 - x_2 \qquad x_1 = (c_1 + c_2 + c_2 t)\, e^{3t}$$
$$\dot{x}_2 = x_1 + 2x_2 \qquad x_2 = (c_1 + c_2 t)\, e^{3t}$$

$$\dot{x}_1 = -x_1 + x_2 + x_3 \qquad x_1(0) = 1 \qquad x_1 = \frac{1}{6} e^{2t} + \frac{1}{3} e^{-t} + \frac{1}{2} e^{-2t}$$
$$\dot{x}_2 = x_1 - x_2 + x_3 \qquad x_2(0) = 0 \qquad x_2 = \frac{1}{6} e^{2t} + \frac{1}{3} e^{-t} - \frac{1}{2} e^{-2t}$$
$$\dot{x}_3 = x_1 + x_2 + x_3 \qquad x_3(0) = 0 \qquad x_3 = \frac{1}{3} e^{2t} - \frac{1}{3} e^{-t}$$

$$\dot{x}_1 = x_2 + x_3 \qquad x_1(0) = -1 \qquad x_1 = -e^{-t}$$
$$\dot{x}_2 = -x_1 + x_3 \qquad x_2(0) = 1 \qquad x_2 = e^{-t}$$
$$\dot{x}_3 = x_1 + x_2 \qquad x_3(0) = 0 \qquad x_3 = 0$$

$\dot{X} = AX \qquad \underbrace{T^{-1}\dot{X}}_{\dot{Y}} = T^{-1}AX = (T^{-1}AT)T^{-1}X = BT^{-1}X$

$\Downarrow$

tj. $\dot{Y} = BY \quad , \quad B = T^{-1}AT$ - dijagonalna matrica

$Y = e^{Bt}C \Rightarrow \quad e^{Bt} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \lambda_1$ i λ_2 jer B dijagonalna matrica

Pod linearnim sistemom s konstantnim koeficijentima naziva se sistem

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}\, x_j + f_i(t) \qquad (i = 1, 2, \dots n) \qquad (1)$$

gdje su a_{ij} realni brojevi ili $a_{ij} \in C$

f_i zadane funkcije

Ako je $f_i(t) \equiv 0$ sistem je <u>homogen</u>

Rješenjem sistema (1) na $\langle a, b\rangle$ naziva se skup funkcija

$$\begin{aligned} x_1 &= \varphi_1(t) \\ x_2 &= \varphi_2(t) \qquad (2) \\ &\vdots \\ x_n &= \varphi_n(t) \end{aligned}$$

određenih i neprekidnih diferencijabilnih funkcija na $\langle a, b\rangle$

Ako funkcije (2) prevode sistem (1) u identitete $\forall t \in \langle a, b\rangle$ tada je (2) rješenje od (1)

Problem iznalaženja rješenja sistema (1) tj. funkcija

$$\begin{aligned} x_1 &= x_1(t) \\ x_2 &= x_2(t) \qquad (3) \\ &\vdots \\ x_n &= x_n(t) \end{aligned}$$

koje zadovoljava početne uvjete

$$x_1(t_0) = x_1^0 \qquad x_2(t_0) = x_2^0 \quad \dots \quad x_n(t_0) = x_n^0 \qquad (4)$$

naziva se Cauchyev problem

Metod rješavanja sistema (1) može se provesti na više načina

(1) svođenje na dif. jed. n-tog reda

$$\frac{dx}{dt} = y+1 \Rightarrow y = \frac{dx}{dt} - 1 \Rightarrow \frac{dy}{dt} = \frac{d^2x}{dt^2}$$

$$\frac{dy}{dt} = x+1$$

$$\frac{d^2x}{dt^2} = x+1 \Rightarrow$$

$$\frac{d^2x}{dt^2} - x - 1 = 0$$

opće rješenje ovo je

$$x = c_1 e^{t} + c_2 e^{-t} - 1 \Rightarrow$$

$$\frac{dx}{dt} = c_1 e^{t} - c_2 e^{-t}$$ ovo u $y = \frac{dx}{dt} - 1 \Rightarrow$

$$y = c_1 e^{t} - c_2 e^{-t} - 1$$

Dakle je opće rješenje

$$x = c_1 e^{t} + c_2 e^{-t} - 1$$
$$y = c_1 e^{t} - c_2 e^{-t} - 1$$

(2) Eulerov metod za homogene sisteme

Razmotrimo taj metod na sistemu od tri jednačine

$$\frac{dx}{dt} = ax + by + cz$$

$$\frac{dy}{dt} = a_1 x + b_1 y + c_1 z \qquad (1)$$

$$\frac{dz}{dt} = a_2 x + b_2 y + c_2 z$$

Rješenje sistema (1) potražimo u obliku

$$x = \lambda e^{rt} \qquad y = \mu e^{rt} \qquad z = \nu e^{rt} \qquad (2)$$

gdje su λ, μ, ν konstantni brojevi

Stavljajući (2) u (1) i kraćeći s e^{rt} dobiva se

$$(a - r)\lambda + b\mu + c\nu = 0$$

$$a_1\lambda + (b_1 - r)\mu + c_1\nu = 0 \qquad (3)$$

$$a_2\lambda + b_2\mu + (c_2 - r)\nu = 0$$

Ovaj sistem ima netrivijalno rješenje samo ako je

$$\Delta = \begin{vmatrix} a - r & b & c \\ a_1 & b_1 - r & c_1 \\ a_2 & b_2 & c_2 - r \end{vmatrix} = 0 \qquad (4)$$

Jedn. dif. (4) naziva se karakterističnom jednadžbom. To je kubna jedn. dif. obzirom na r

a) Neka su r_1, r_2, r_3 korjeni karakteristične jednadžbe realni i različiti

Stavljajući u (3) mjesto r vrijednost r_1 i rješivši isti **sistem** dobiva se $\lambda_1 \; \mu_1 \; \nu_1$

analogno stavljajući u (3) mjesto r vrijednost r_2 dobiva se $\lambda_2 \; \mu_2 \; \nu_2$ i konačno $r \to r_3 \Rightarrow$ λ_3, μ_3, ν_3

Dakle tim uređenim trojkama $\Rightarrow$

$r \to r_1 \Rightarrow$ $(\lambda_1, \mu_1, \nu_1)$

$x_1 = \lambda_1 e^{r_1 t}$

$y_1 = \mu_1 e^{r_1 t}$

$z_1 = \nu_1 e^{r_1 t}$

$r \to r_2 \Rightarrow$ $(\lambda_2, \mu_2, \nu_2)$

$x_2 = \lambda_2 e^{r_2 t}$

$y_2 = \mu_2 e^{r_2 t}$

$z_2 = \nu_2 e^{r_2 t}$

$r \to r_3 \Rightarrow$ $(\lambda_3, \mu_3, \nu_3)$

$x_3 = \lambda_3 e^{r_3 t}$

$y_3 = \mu_3 e^{r_3 t}$ (5)

$z_3 = \nu_3 e^{r_3 t}$

Dakle je opće rješenje od (1)

$$\begin{aligned} x &= C_1 x_1 + C_2 x_2 + C_3 x_3 \\ y &= C_1 y_1 + C_2 y_2 + C_3 y_3 \qquad (6) \\ z &= C_1 z_1 + C_2 z_2 + C_3 z_3 \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} C \\ \\ \end{bmatrix}$$

Primjer:

$$\frac{dx}{dt} = 3x - y + z$$

$$\frac{dy}{dt} = -x + 5y - z \qquad (7)$$

$$\frac{dz}{dt} = x - y + 3z$$

Karakt. jed.

$$(8) \qquad \begin{vmatrix} 3-r & -1 & 1 \\ -1 & 5-r & -1 \\ 1 & -1 & 3-r \end{vmatrix} = 0$$

$$r^3 - 11r^2 + 36r - 36 = 0$$

$r_1 = 2$
$r_2 = 3$
$r_3 = 6$

$\lambda_1 = 1$ $\qquad \mu_1 = 0$ $\qquad \nu_1 = -1$

$\lambda_2 = 1$ $\qquad \mu_2 = 1$ $\qquad \nu_2 = 1$

$\lambda_3 = 1$ $\qquad \mu_3 = -2$ $\qquad \nu_3 = 1$

$x_1 = e^{2t}$ $\qquad y_1 = 0$ $\qquad z_1 = -e^{2t}$

$x_2 = e^{3t}$ $\qquad y_2 = e^{3t}$ $\qquad z_2 = e^{3t}$ $\qquad (9)$

$x_3 = e^{6t}$ $\qquad y_3 = -2e^{6t}$ $\qquad z_3 = e^{6t}$

Opće rješenje je

$$x = c_1 e^{2t} + c_2 e^{3t} + c_3 e^{6t}$$

$$y = c_2 e^{3t} - 2c_3 e^{6t} \qquad (10)$$

$$z = -c_1 e^{2t} + c_2 e^{3t} + c_3 e^{6t}$$

b) Neka su korijeni karakteristične jednadžbe kompleksni

primjer
$$\frac{dx}{dt} = x - 5y$$
$$\frac{dy}{dt} = 2x - y \qquad (11)$$

$$x = \lambda e^{rt} \qquad y = \mu e^{rt}$$
$$\frac{dx}{dt} = \lambda r e^{rt} \qquad \frac{dy}{dt} = \mu r e^{rt}$$

$$(1-r)\lambda - 5\mu = 0$$
$$2\lambda - (1+r)\mu = 0 \qquad (12)$$

Karakteristična jed.

$$\begin{vmatrix} 1-r & -5 \\ 2 & -1-r \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{matrix} r_1 = 3i \\ r_2 = -3i \end{matrix}$$

stavljajući $r_1 = 3i$ u (12) dobiva se

$$(1-3i)\lambda_1 - 5\mu_1 = 0$$
$$2\lambda_1 - (1+3i)\mu_1 = 0$$

Ove dvije jednadžbe su linearno zavisne jer im je determinanta jednaka nuli.

Sada se ovdje treba formirati novi fundamentalni sistem:

$$(1-3i)\lambda_1 = 5\mu_1$$

$$\frac{\lambda_1}{\mu_1} = \frac{5}{1-3i} \Rightarrow$$ Uzimajući

$$\lambda_1 = 5 \qquad \mu_1 = 1-3i$$

dobivamo prvi skup rješenja

$$x_1 = 5e^{3it} \qquad y_1 = (1-3i)e^{3it} \qquad (13)$$

Analogno stavljajući u (12) $r_2 = -3i$ dobiva se

$$(1+3i)\lambda_2 - 5\mu_2 = 0$$
$$2\lambda_2 - (1-3i)\mu_2 = 0$$

$$\begin{vmatrix} 1+3i & -5 \\ 2 & -(1-3i) \end{vmatrix} = -(1-9i^2)+10 = 0$$

Jer ove dvije jednadžbe su međusobno linearno zavisne pa je

$$\frac{\lambda_2}{\mu_2} = \frac{5}{1+3i} \Rightarrow \begin{array}{l} \lambda_2 = 5 \\ \mu_2 = 1+3i \end{array} \qquad \left.\begin{array}{l} x_2 = 5e^{-3it} \\ y_2 = (1+3i)e^{-3it} \end{array}\right\} (14)$$

Prelazeći na novi fundamentalni sistem rješenja

$$\left.\begin{array}{ll} \tilde{x}_1 = \dfrac{x_1 + x_2}{2} & \tilde{x}_2 = \dfrac{x_1 - x_2}{2i} \\ \tilde{y}_1 = \dfrac{y_1 + y_2}{2} & \tilde{y}_2 = \dfrac{y_1 - y_2}{2i} \end{array}\right\} \quad (15)$$

i služeći se Eulerovim formulama

$$e^{\pm\alpha i t} = \cos\alpha t \pm i\sin\alpha t$$

gdje je α realan broj iz (14) i (15) $\Rightarrow$

$$\tilde{x}_1 = 5\cos 3t \qquad \tilde{x}_2 = 5\sin 3t$$
$$\tilde{y}_1 = \cos 3t + 3\sin 3t \qquad \tilde{y}_2 = \sin 3t - 3\cos 3t$$

Opće rješenje je

$$x = c_1 \tilde{x}_1 + c_2 \tilde{x}_2 = 5 c_1 \cos 3t + 5 c_2 \sin 3t$$

(16)

$$y = c_1 \tilde{y}_1 + c_2 \tilde{y}_2 = c_1 (\cos 3t + 3 \sin 3t) + c_2 (\sin 3t - 3 \cos 3t)$$

Ako karakteristična jednadžba ima neki korijen k_i kratnosti s_i tada njemu korespodentno rješenje ima oblik

$$\begin{aligned} x_{1i} &= P_{1i}(t)\, e^{k_i t} \\ x_{2i} &= P_{2i}(t)\, e^{k_i t} \\ &\vdots \\ x_{ni} &= P_{ni}(t)\, e^{k_i t} \end{aligned} \qquad (17)$$

gdje je $P_{ji}(t)$ polinom $s_i - 1$ stepena s neodređenim koeficijentima koje izračunavamo da ovo stavimo u zadani sistem jednadžbi

Riješiti sistem

$$\frac{dx}{dt} = 2x + y \qquad 18_1$$

$$\frac{dy}{dt} = 4y - x \qquad 18_2$$

Karakt. jed.

$$\begin{vmatrix} 2-r & 1 \\ -1 & 4-r \end{vmatrix} = 0$$

$$r^2 - 6r + 9 = 0$$

$$r_1 = r_2 = 3$$

Rješenje treba tražiti u obliku

$$x = (\lambda_1 + \mu_1 t)\, e^{3t}$$
$$y = (\lambda_2 + \mu_2 t)\, e^{3t} \qquad (19)$$

Stavljajući (19) u (18_1) dobiva se

$$3(\lambda_1 + \mu_1 t) + \mu_1 = 2(\lambda_1 + \mu_1 t) + (\lambda_2 + \mu_2 t) \qquad (20)$$

Izjednačavajući koeficijente uz iste potencije t ⇒

$$3\lambda_1 + \mu_1 = 2\lambda_1 + \lambda_2$$
$$3\mu_1 = 2\mu_1 + \mu_2 \qquad 21$$

$$\lambda_2 = \lambda_1 + \mu_1$$
$$\mu_2 = \mu_1 \qquad (22)$$

λ_1, μ_1 su proizvoljne konst. nazovimo ih C_1 i C_2 (λ_1, μ_1)

$$x = (C_1 + C_2 t)\, e^{3t}$$
$$y = (C_1 + C_2 + C_2 t)\, e^{3t}$$

Teorija stabilnosti

Neka je zadan sistem diferencijalnih jednadžbi

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots x_n, t) \qquad (1)$$

Za rješenje $\varphi_i(t)$ $i = 1, 2, \ldots n$ sistema (1) reći ćemo da je stabilno u smislu Ljapunova ako za svako $\varepsilon > 0$ postoji $\delta > 0$ takvo da za svako rješenje

$x_i(t)$ $i = 1, 2, \ldots, n$ sistema ~~je~~ (1)

početni uvjeti kojeg zadovoljavaju uvjetima

$$|x_i(t_0) - \varphi_{i0}| < \delta \qquad i = 1, 2, \ldots n \qquad (2)$$

vrijedi nejednakost

$$|x_i(t) - \varphi_i(t)| < \varepsilon \qquad i = 1, 2, \ldots, n \qquad (3)$$

$\forall\, t \geq t_0$, gdje je $\varphi_{i0} = \varphi_i(t_0)$ $i = 1, 2, 3, \ldots n$ početni uvjet

Ako za neki proizvoljno mali $\delta > 0$ postoji bar jedno rješenje $x_i(t)$ $i = 1, 2, \ldots, n$ takvo da nejednakost (3) ne vrijedi tada za rješenje $\varphi_i(t)$ kažemo da je nestabilno

Ako pored nejednakosti (3) uz uvjet (2) vrijedi i ~~nejednak~~ jednakost

$$\lim_{t \to \infty} |x_i(t) - \varphi_i(t)| = 0 \qquad i = 1, 2, \ldots n \qquad (4)$$

tada se rješenje $\varphi_i(t)$ naziva asimptotski stabilno